FIBONACCI SEQUENCES AND GROUP THEORY

by

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STATEMENT

The work in this thesis is my own unless otherwise indicated.

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ABSTRACT

Three topics are studied in this thesis.

The first concerns a sequence of elements of a group called a Fibonacci sequence. If two elements a and b of a group are given then a Fibonacci sequence on G is a sequence: $a, b, ba, bab, bab^2a, \ldots$, where every three consecutive terms have the form u, v, vu. We study the periodicity of Fibonacci sequences on groups. Some information about the freest two generator group to have a Fibonacci sequence of a given length is obtained - mainly by the study of certain factor groups. Our results on this topic are mostly of a combinatorial nature.

The second is on T-systems of a group. For an n-generator group these are certain equivalence classes of the generating n-tuples. Fibonacci sequences are used in the determination of the T-systems of several well known finite groups. We prove that a two generator group with an abelian normal subgroup such that the quotient group by this subgroup is infinite cyclic is a metabelian group with one T-system of generating pairs. The group $\text{gp}\{a, b;\ b^{-1}a^2b = a^3\}$ is shown to have an infinite number of T-systems of generating pairs. A presentation is associated with a representative generating pair from each of these T-systems.

The third topic deals with an isomorphism problem for two generator one relator groups which are an extension of a free group of finite rank by a free group of rank one. The problem studied is that of deciding when two one relator presentations of groups of this type determine isomorphic groups. It is found to be sufficient to consider certain "standard" presentations for these groups. Two sets of isomorphism criteria are established which make it possible to distinguish these groups modulo their second derived group. When the groups considered are an extension of a free group of rank two by a free group of rank one the problem is solved completely and the groups determined.

CONTENTS

INTRODUCTION

Three topics are considered in this thesis: an analogue in group theory of the Fibonacci sequence in number theory, a certain equivalence class of generating pairs of a group called T-systems, and an isomorphism problem for a class of two generator one relator groups which are cyclic extensions of a free group.

Fibonacci sequences on a two generator group G are formed by taking two generators a and b of G and writing the following sequence of elements of G : $a, b, ba, bab, bab^2a, \ldots$, where for each three consecutive terms u, v and w we stipulate that $w = vu$.

The study of Fibonacci sequences of integers modulo some natural number is effectively the study of Fibonacci sequences on a cyclic group. For properties of these sequences we refer to D.D. Wall [23]. To our knowledge there is no literature which deals with Fibonacci sequences on non-abelian groups apart from a paper by F. Roesler [20].

One of the most immediate questions which can be asked concerning a Fibonacci sequence on a group is if it is periodic, and if so what the least period (called the length) is. As a consequence most of our results on Fibonacci sequences in Chapter 1 are concerned with periodicity. For finite groups we can show, and it is almost self-evident, that every Fibonacci sequence is periodic. However, there are infinite groups with periodic Fibonacci sequences as is seen, for example, in Lemma 1.3.7.

To find which groups have periodic Fibonacci sequences it is convenient to define for each non-negative integer l a certain group G_l . This group G_l is shown in Lemma 1.2.1 to be the freest two generator group having a Fibonacci sequence of length l. For small values of l, the groups G_l are determined (§1.3) fairly easily: G_3 is the quaternion group of order 8, G_4, G_5 and G_7 are cyclic of orders 5, 11 and 29 respectively, and G_6 is a certain infinite metabelian group. When l is greater than 7, however, this becomes less straightforward and we resort to looking at factor groups of G_l.

Clearly the first factor group to be considered is the commutator quotient group G_l/G_l' which is the freest two generator abelian group with a Fibonacci sequence of length l. The terms of a Fibonacci sequence on an abelian group can be written in a particularly simple form (Lemma 1.4.1), and because of this we can give (in Corollary 1.4.3) a presentation of G_l/G_l' which provides a useful description of the group.

The next factor groups which we consider (§1.5) are the groups $G_l/[G_l', G_l]$, which are nilpotent of class 2, and certain groups H_l which have a Fibonacci sequence of length l such that the squares of the initial two terms are central. It is convenient to handle separately the cases when l is odd and when l is even.

Firstly when l is odd both H_l and $G_l/[G_l', G_l]$ define the same group which has been completely determined by F. Roesler in [20]. Incidentally we are able to answer one of the problems considered by Roesler in his work on these groups with an example, due to R.M. Bryant, of a finite epimorphic image of G_{15} which is not nilpotent of class 2.

On the other hand, when l is even, H_l and $G_l/[G_l', G_l]$ do not define the same group. We prove in Theorem 1.5.3 that H_l is abelian when l is not divisible by 3, and when l is divisible by 3 that H_l is an extension (not necessarily split) of an abelian group (which is infinite in general) by the quaternion group of order 8. Also, when l is even the groups $G_l/[G_l', G_l]$ are finite and if l is even and divisible by neither 3 nor 5 we establish in Theorem 1.5.6 a fairly simple presentation for these groups.

Rather surprisingly, a problem attributed to E. Netto and studied by A.-L. Arrhenius-Wold in [1] has a connection with the groups G_l we have been discussing. The problem stated in [1] is to find those integers l for which the group $\mathrm{gp}(a, b;\ a^l = 1, ba = a^2b^2)$ is finite and non-trivial. We are able to show that this group is the split extension of G_l by a certain automorphism of order l thus reducing the problem to determining the groups G_l. A large part of [1] is devoted to the study of the group $\mathrm{gp}(a, b;\ a^l = b^l = 1, ba = a^2b^2)$. In §1.4 we show that this group is the split extension of G_l/G_l' by a certain automorphism of order l and is therefore determined by our results on the groups G_l/G_l'.

The problem of finding the length (if the sequence is periodic) of a Fibonacci sequence on a group which is given in terms of generators and defining relations is discussed in §1.6. Of particular interest is the investigation of the periodicity of Fibonacci sequences on two generator groups of some given exponent. For groups of exponent 2, 3 and 4 we find sequence lengths 3, 8 and 12 respectively. Some of our more detailed results in Theorem 1.6.1 and Theorem 1.6.4 are: the groups $\mathrm{gp}(a, b;\ a^3 = b^3 = (ab)^3 = 1)$ and

$\mathrm{gp}(a, b;\ a^4 = b^2 = (ab)^4 = 1)$ have aperiodic Fibonacci sequences, and the groups $\mathrm{gp}(a, b;\ a^3 = b^3 = (ab)^3 = (a^{-1}b)^{3k} = 1)$ and $\mathrm{gp}(a, b;\ a^4 = b^2 = (ab)^4 = (a^2b)^{2k} = 1)$ have Fibonacci sequences of length $8k$ and $6k$ respectively, for any positive integer k.

For the most part this thesis deals with two generator groups. The reason for this is our interest in Chapter 1 in calculating Fibonacci sequences from the generating pairs of a group. Now the consecutive pairs of elements of a Fibonacci sequence can be shown to also be generating pairs of the group. Our interest in the second topic studied in this thesis stemmed from investigations of the relation between presentations associated with generating pairs formed from consecutive elements in a Fibonacci sequence, and similar considerations.

We now describe briefly the main concepts used in Chapter 2. Let G be an arbitrary two generator group and B its automorphism group. Let F be the free group generated by x and y with automorphism group A. For each β in B we define an action on generating pairs (a, b) of G by setting $(a, b)\beta = (a\beta, b\beta)$. Each element of A is defined to act on generating pairs of G in a similar way to which Nielsen transformations are defined to act on generating pairs of F; in other words as a "free isomorphism" (see [12], p. 170). The transitivity sets of generating pairs of G under the action of the elements of A, B and $\{A, B\}$ are called A-classes, B-classes and T-systems respectively.

T-systems were defined and studied by B.H. Neumann and Hanna Neumann in [14] where they were studied in connection with certain characteristic subgroups of free groups. Subsequently several questions related to T-systems raised in [14] and [15] were answered by M.J. Dunwoody in [6] and [7]. We refer to [19] and §3.6 of [12] for the related concepts of "free isomorphism" and "free automorphism". For more recent work connected with T-systems we refer to [8], [13] and [18].

Now it is shown in [14] that a T-system can be thought of as a transitivity set of B-classes under the action of representatives of A modulo I the group of inner automorphisms of F. Moreover A is generated modulo I by the elements φ and μ defined by $x\varphi = y$, $y\varphi = yx$ and $x\mu = y$, $y\mu = x$. However, in Lemma 1.1.1 we show that the sequence $x, x\varphi, x\varphi^2, \ldots$ is the Fibonacci sequence on F with initial terms x and y. Also if (a, b) is a generating pair of a group G then $(a, b)\varphi^k$ is the generating pair of G which has entries the $(k+1)$st and $(k+2)$nd terms of the Fibonacci sequence with initial elements a and b. These facts are used in the appendix to this thesis to calculate the T-systems of several well known finite groups.

In §2.3 we discuss the importance of knowing the T-systems of a group particularly in relation to a classification of the presentations which a group has.

It is known that finitely generated abelian groups have only one T-system, but may have more than one A-class. In [15] B.H. Neumann found an example of a finite nilpotent group with more than one T-system. This result was then sharpened by M.J. Dunwoody in [6] who found examples of finite p-groups of class 2 having arbitrary finite numbers of T-systems of generating pairs.

In contrast to these results we show in Theorem 2.5.1 that a two generator group G with an abelian normal subgroup N such that G/N is an infinite cyclic group is a metabelian group with only one

T-system of generating pairs. Examples of groups of this kind are the groups $\mathrm{gp}(a, b;\ b^{-1}ab = a^r)$ for every integer r (Theorem 2.5.2). We mention a result of M.J. Dunwoody (Theorem 4.10 of [7]) that a two generator metabelian group whose commutator quotient group is free abelian of rank 2 has only one T-system of generating pairs. Lastly we show in Theorem 2.5.3 that splitting metacyclic groups also have only one T-system of generating pairs.

Our main result in Chapter 2 (Theorem 2.6.6) is to show that the group $G = \mathrm{gp}(a, b;\ b^{-1}a^2b = a^3)$ has an infinite number of T-systems of generating pairs. In Theorem 2.6.4 we find presentations associated with certain representative generating pairs in these T-systems.

The group $G = \mathrm{gp}(a, b;\ b^{-1}a^2b = a^3)$ was shown to be a non-Hopf group in [2], and according to [12] (p. 416) G is "the simplest example of a non-Hopfian finitely presented group". We construct an infinite ascending chain of non-Hopf kernels,

$N_n = \left\langle [a, a^b], \left[a, a^{b^n}\right] \right\rangle^G$ for $n = 1, 2, \dots$, each contained in the second derived group G'' . It is interesting to note that the group $G \,/\, \bigcup_{n=1}^{\infty} N_n$ is a group isomorphic to G/G'' which can be shown (by one of our results mentioned above) to have only one T-system.

As a byproduct we find that G has two generating pairs associated with one relator presentations which lie in different T-systems. Thus G provides a further counterexample to a certain conjecture of W. Magnus. We discuss Magnus's conjecture, its relation to T-systems, and other counterexamples due to J. McCool and A. Pietrowski in §2.4.

The last chapter in this thesis is concerned with the class $X(r)$ of two generator one relator groups which are extensions of a free group of finite rank r by a free group of rank one. We prove some isomorphism results to do with the one relator presentations of these groups.

In [19] E.S. Rapaport showed that there are only three non-isomorphic groups in $X(2)$ with commutator quotient groups free of rank one. She also showed that these groups have one A-class of generating pairs associated with one relator presentations. One of the groups, the group $\mathrm{gp}\left(a, b;\ ba = a^2b^2\right)$, is of interest in connection with a problem of Netto mentioned earlier and the other two are the trefoil knot group and the group of Listing's knot. It is interesting to note that M.J. Dunwoody and A. Pietrowski [8] have shown that the trefoil knot group has an infinite number of T-systems of generating pairs. Only one of these, however, contains generating pairs associated with one relator presentations.

An essential result in [19] (Lemma 1) states that for a group in $X(2)$ with commutator quotient group free of rank one every generating pair associated with a one relator presentation belongs to the same A-class as a generating pair associated with a certain standard presentation (the proof is attributed to W. Magnus). In Theorem 3.1.1 this result is extended to the groups in $X(r)$. Here a standard presentation is a presentation $\mathrm{gp}\left(a, b;\ a^{b^r} = ua^{\varepsilon}u'\right)$, where $\varepsilon = \pm 1$ and u, u' belong to the (free) group generated by $a^b, a^{b^2}, \ldots, a^{b^{r-1}}$. It is shown in Lemma 3.1.2 that any group with a standard presentation belongs to the class $X(r)$ for some r.

The aim of our work with these groups is to find when two standard presentations define isomorphic groups. Also it is of interest to know whether or not the generating pairs associated with standard presentations lie in the same A-class (or even the same T-system).

The first criteria for the isomorphism of two groups in $X(r)$ are obtained in §3.1. We impose the restriction that the commutator quotient group of each group considered should not be free abelian of rank 2. For such a group G Lemmas 3.1.2 and 3.1.3 show that G has a free characteristic subgroup H such that G/H is free of rank one. In Lemma 3.1.4 we calculate the coefficients of the characteristic polynomial of the action of a generator of G/H on H/H'. The coefficients are easily obtained from a standard presentation of G. In Theorem 3.1.5 we give the relationship between these coefficients for isomorphic groups in $X(r)$ whose commutator quotient groups are not free abelian of rank two.

At this stage the information obtained so far is used to characterize the groups in $X(2)$. Thus in Theorem 3.2.2 it is shown that every group in $X(2)$ has one A-class of generating pairs which are associated with one relator presentations. Moreover Corollary 3.2.3 states that in $X(2)$ there are: two non-isomorphic groups whose commutator quotient group is free abelian of rank two, three non-isomorphic groups whose commutator quotient group is free of rank one, and three non-isomorphic groups for each positive integer s greater than 1 each of whose commutator quotient group is the direct product of a cyclic group of order s and an infinite cyclic group.

Unfortunately the method by which Theorem 3.1.5 was used to determine the groups in $X(2)$ is not applicable when considering the groups in $X(r)$ for r greater than 2. Thus in §3.3 we study further how to decide when two standard presentations of groups in

$\mathfrak{X}(r)$ define isomorphic groups. We impose the restriction that the commutator quotient groups of the groups considered should not be torsion free. The results obtained determine the standard presentations of isomorphic groups in $\mathfrak{X}(r)$ modulo their second derived groups.

If C and D are isomorphic groups in $\mathfrak{X}(r)$ then there is an isomorphism induced between C/C' and D/D' and thus a correspondence between the characteristic polynomials of the action of elements of C/C' on C'/C'' and of D/D' on D'/D''. The basic idea of our work is to use this correspondence between characteristic polynomials to find the relationship between the standard presentations. For our purpose it is enough to find the relationship between certain "standard vectors" which are easy to calculate from standard presentations. Theorem 3.3.1 states the exact relationship holding between standard vectors obtained from the standard presentations of isomorphic groups in $\mathfrak{X}(r)$ whose commutator quotient groups are not torsion free.

In order to prove this theorem we analyse the action of certain elements of a group in $\mathfrak{X}(r)$ on the group G'/G''. A certain G-invariant subgroup W of G'/G'' is found and it is enough to consider the action of elements of G on $V = (G'/G'')/W$. Now V, which is a free abelian group and therefore a free $\mathbb{Z}$-module, can be tensored over the integers with the complex numbers. It is shown that the resulting module has a decomposition into certain G-invariant submodules on which the characteristic polynomials of the action of elements of G are easy to calculate. The correspondence between these characteristic polynomials in isomorphic groups is then determined, and by equating the coefficients of equal characteristic polynomials the information required for the proof of the theorem is found.

In the final section (§3.4) of Chapter 3 we give a simple example of two groups in $X(3)$ which are indistinguishable by the results of §3.2 but which are shown to be not isomorphic by the results of §3.3.

NOTATIONS AND PRELIMINARIES

We list here some notations used in the thesis.

$\mathrm{sbgp}(a, b, c, \ldots)$	:	the subgroup generated by elements $a, b, c, \ldots$.
$\langle a, b, c, \ldots \rangle^G$	:	the smallest normal subgroup of a group G generated by the set $\{a, b, c, \ldots\}$.
a^b	:	$b^{-1}ab$.
$[a, b]$	:	$a^{-1}b^{-1}ab$.
$[a, rb]$	:	$[a, 1b] = [a, b]$ and $[a, rb] = [a, (r-1)b]$ when $r > 1$.
$[A, B]$	:	the group generated by $\{[a, b]; a \in A, b \in B\}$.
$Z(G)$	:	the centre of a group G .
G'	:	$[G, G]$.
G''	:	$[G', G']$.
Z	:	the ring of integers.
Z_s	:	the ring of integers modulo s .
C	:	the field of complex numbers.
$\|s\|$	:	the modulus of an integer s .

This thesis is concerned with groups mainly from the point of view of their generators and defining relations. For fundamental concepts from group theory and as a basic reference we refer to [12]. In general if a term or concept is undefined in this thesis it can be found in [12].

We differ a little from [12] as regards notation. In particular it will be convenient to distinguish (more than is done in [12]) two presentations of a group; an abstract presentation and a presentation in terms of generators and defining relations. The reason for this approach is that we shall quite often wish to compare on the one hand different presentations of a group, and on the other the different generating sets of a group.

A *group presentation* $(x, y, z, \ldots; r, s, t, \ldots)$ is an object consisting of a set of distinct symbols $x, y, z, \ldots$ and a set of elements $r, s, t, \ldots$ in the free group F on the set $\{x, y, z, \ldots\}$. The elements $r, s, t, \ldots$ are called relators.

Let G be a group and suppose that $a, b, c, \ldots$ are elements which generate G. There is an epimorphism ν, called the canonical epimorphism of F onto G with $x\nu = a$, $y\nu = b$, $z\nu = c$, ... The group G is said to have a presentation $(x, y, z, \ldots; r, s, t, \ldots)$ if the kernel of ν is equal to $\langle r, s, t, \ldots\rangle^F$ (we sometimes write $R(a, b, c, \ldots)$ for $\langle r, s, t, \ldots\rangle^F$ to emphasise its connection with the generating tuple $(a, b, c, \ldots)$). We speak of the generating tuple $(a, b, c, \ldots)$ as being associated with a presentation $(x, y, z, \ldots; r, s, t, \ldots)$ of G.

Let $\overline{F}$ denote the free group generated by $a, b, c, \ldots$, and let η be the identity isomorphism of F onto $\overline{F}$ with $x\eta = a$, $y\eta = b$, $z\eta = c$, ... Then, we write

$$G = \mathrm{gp}(a, b, c, \ldots; r\eta = 1, s\eta = 1, t\eta = 1, \ldots)$$

and speak of G as being *presented with generators and defining relations*.

Two sets of relators $\{r, s, t, \ldots\}$ and $\{\overline{r}, \overline{s}, \overline{t}, \ldots\}$ are said to be *equivalent* if $\langle r, s, t, \ldots\rangle^F = \langle \overline{r}, \overline{s}, \overline{t}, \ldots\rangle^F$.

Clearly if $(a, b, c, \ldots)$ is a generating tuple of a group G associated with a presentation $(x, y, z, \ldots; r, s, t, \ldots)$ and the sets of relators $\{r, s, t, \ldots\}$ and $\{\overline{r}, \overline{s}, \overline{t}, \ldots\}$ are equivalent then $(a, b, c, \ldots)$ is also associated with the presentation $(x, y, z, \ldots; \bar{r}, \bar{s}, \bar{t}, \ldots)$.

In practice we shall not refer explicitly to the isomorphism η above. Thus if G is a group with a generating pair (a, b) associated with a presentation $(x, y; x^{-5}yx^{5}y)$ of G then we write $G = \mathrm{gp}(a, b; ba^{5}b = a^{5})$.

We state here the following commutator identities (see, for example, 33.34 of [17]);

$$[uv, wz] = [u, z]^{v}[v, z][u, w]^{vz}[v, w]^{z} ,$$

$$[u^{-1}, w] = [w, u]^{u^{-1}} , \quad [u, w^{-1}] = [w, u]^{w^{-1}} ,$$

$$[u^{-1}, w^{-1}] = [u, w]^{w^{-1}u^{-1}} .$$

As a consequence of these identities it can be shown that if G is a two generator group with a generating pair (a, b) then $G' = \langle [a, b] \rangle^{G}$.

A group G is said to be nilpotent of class 2 if $[G', G] = 1$, and metabelian if $[G', G'] = 1$ where 1 stands here for the trivial group.

Clearly if G is a two generator group with a generating pair (a, b) then G is nilpotent of class 2 if and only if $[a, b]$ is central in G.

In Chapter 3 we shall require the concept of exponent sum. This definition comes from p. 76 of [12].

Let w be a word in the elements $a_1, a_2, \ldots, a_n$ and

$$w = a_{i_1}^{\alpha_1} a_{i_2}^{\alpha_2} \ldots a_{i_n}^{\alpha_n}$$

where α_j are integers and $i_j = 1, 2, \ldots, n$. Then the *exponent* *sum* of w on a_j is the integer $\sigma_{a_j}(w) = \sum_{i_k=j} \alpha_k$.

Let G be a group with a normal subgroup N . We define an action of elements of G on N/N' by setting

$$(hN')^g = h^g N' ,$$

for every g in G and h in N . It is not hard to check that the elements of G , with this action on the abelian group N/N' , induce automorphisms of N/N' . Moreover, if k belongs to N then g and gk induce the same automorphism of N/N' ; for

$$\begin{aligned}(hN')^{gk} &= h^{gk}N' \\ &= h^g h^{-g} h^{gk} N' \\ &= h^g N' \text{ as } h^{-g}(h^g)^k \text{ belongs to } N' , \\ &= (hN')^g .\end{aligned}$$

In this way we may regard N/N' as a $\mathbb{Z}G$-module, or even as a $\mathbb{Z}G/N$-module. We will sometimes write N/N' additively. In this case if h belongs to N , $s_1, \ldots, s_r$ are integers, and $g_1, \ldots, g_r$ elements of G then by $s_1(hN')^{g_1} + \ldots + s_r(hN')^{g_r}$ (sometimes written as $(hN')^{s_1g_1+\ldots+s_rg_r}$) is meant the element $(hN')^{s_1g_1} \ldots (hN')^{s_rg_r}$.

CHAPTER I

FIBONACCI SEQUENCES ON GROUPS

1.1 Preliminaries

Let F be the free group of rank two freely generated by x and y. For each integer n we define a word w_n in F called the *n-th Fibonacci word.*

Let $w_1 = x$, $w_2 = y$ and

$$w_n = \begin{cases} w_{n-1}w_{n-2} & \text{if } n > 2 \\ w_{n+1}^{-1}w_{n+2} & \text{if } n < 1 . \end{cases}$$

For example $w_3 = yx$, $w_4 = yxy$, $w_5 = yxy^2x$, $w_6 = yxy^2xyxy$, $w_7 = yxy^2xyxy^2xy^2x$, and $w_0 = x^{-1}y$, $w_{-1} = y^{-1}x^2$, $w_{-2} = x^{-2}yx^{-1}y$.

If G is an arbitrary group and a, b are elements of G then there is a canonical epimorphism ν of F onto the subgroup of G generated by a and b such that $x\nu = a$ and $y\nu = b$. We denote for each integer n, the element $w_n\nu$ of G by $f_n(a, b)$. The sequence of elements $f_n(a, b)$, where n is an integer, denoted by $f(a, b)$ is called the Fibonacci sequence on G with initial values a and b.

In a particular situation where the initial elements a and b are understood we abbreviate $f_n(a, b)$ to f_n and $f(a, b)$ to f. We note that since ν is a homomorphism the sequence f enjoys the property:

if n is any integer then $f_n = f_{n-1}f_{n-2}$ and $f_n = f_{n+1}^{-1}f_{n+2}$.

Now the formation of Fibonacci sequences is local in the sense that given two elements a and b as initial elements of the sequence the elements $f_n(a, b)$, for every integer n, belong to the subgroup generated by a and b. Consequently, we will always consider Fibonacci sequences on a group generated by the initial elements.

Suppose now that G is the additive group of integers. If $a = b = 1$ then the sequence $f(1, 1)$ is the Fibonacci sequence of number theory (actually the ordinary Fibonacci sequence whose index set is extended to include non-positive integers, see 10.14.5 of [11]). We will reserve the letters $u(n)$ for the elements $f_n(1, 1)$ of this sequence noting that the formulae above become $u(n) = u(n-1) + u(n-2)$ and $u(n) = u(n+2) - u(n+1)$ for any integer n. There is also a sequence, called the Lucas sequence, which is the sequence $f(1, 3)$ and we reserve the letters $v(n)$ for the elements $f_n(1, 3)$ of this sequence. We refer to [11] and to [22] for elementary properties of these sequences.

There is a sequence of words w'_n in F very closely related to the sequence of Fibonacci words w_n.

Let $w'_1 = x$, $w'_2 = y$ and

$$w'_n = \begin{cases} w'_{n-2}w'_{n-1} & \text{if } n > 2 \\ \\ w'_{n+2}\left(w'_{n+1}\right)^{-1} & \text{if } n < 1 . \end{cases}$$

Now there is an automorphism ξ of F with $x\xi = x^{-1}$ and $y\xi = y^{-1}$. It can be readily checked that $\left(w_n\xi\right)^{-1} = w'_n$ for each integer n. We will continue to use the words w_n but we remark that similar considerations would also apply if we were to use the words w'_n .

We concern ourselves now with some elementary observations about Fibonacci sequences and words. The first deals with a certain "Fibonacci" automorphism of F .

1.1.1 LEMMA. *There is an automorphism* φ *of* F *with* $x\varphi = y$ *and* $y\varphi = yx$. *Further if* k *is any integer then* $x\varphi^k = w_{k+1}$ *and* $y\varphi^k = w_{k+2}$.

Proof. As y and yx generate F it is clear that there is such an automorphism φ . We prove the second assertion by induction on k . Since $y = x\varphi$ it suffices to show $x\varphi^k = w_{k+1}$ for any integer k .

When $k = 0$ we have $w_1 = x$ by definition.

(a) Assume k is a positive integer and that the assertion is true for every non-negative integer less than k .

Then

$$
\begin{aligned}
x\varphi^k &= \left(x\varphi^{k-1}\right)\varphi \\
&= w_k\varphi && \text{by induction hypothesis,} \\
&= \left(w_{k-1}w_{k-2}\right)\varphi && \text{by definition,} \\
&= \left(w_{k-1}\varphi\right)\left(w_{k-2}\varphi\right) \\
&= \left(\left(x\varphi^{k-2}\right)\varphi\right)\left(\left(x\varphi^{k-3}\right)\varphi\right) && \text{by induction hypothesis,} \\
&= \left(x\varphi^{k-1}\right)\left(x\varphi^{k-2}\right) \\
&= w_k w_{k-1} && \text{by induction hypothesis,} \\
&= w_{k+1} && \text{by definition,}
\end{aligned}
$$

and this completes the induction step and the proof for non-negative integers k .

(b) Assume k is a negative integer and that the assertion is true for every non-positive integer greater than k . Then

$$
\begin{aligned}
x\varphi^k &= \left(x\varphi^{k+1}\right)\varphi^{-1} \\
&= w_{k+2}\varphi^{-1} && \text{by induction hypothesis,} \\
&= \left(w_{k+3}^{-1}w_{k+4}\right)\varphi^{-1} && \text{by definition,} \\
&= \left(w_{k+3}\varphi^{-1}\right)^{-1}\left(w_{k+2}\varphi^{-1}\right) \\
&= \left(x\varphi^{k+2}\varphi^{-1}\right)^{-1}\left(x\varphi^{k+3}\varphi^{-1}\right) && \text{by induction hypothesis,} \\
&= \left(x\varphi^{k+1}\right)^{-1}\left(x\varphi^{k+2}\right) \\
&= w_{k+2}^{-1}w_{k+3} && \text{by induction hypothesis,} \\
&= w_{k+1} && \text{by definition,}
\end{aligned}
$$

and this completes the induction step and the proof for non-positive integers k . //

From Lemma 1.1.1 we have that φ^k is an automorphism of F so that $\left(w_{k+1}, w_{k+2}\right)$ is a generating pair of F for any integer k . Thus if ν denotes the canonical epimorphism of F onto a group G generated by a and b with $x\nu = a$ and $y\nu = b$ then, since $w_{k+1}\nu = f_{k+1}$ and $w_{k+2}\nu = f_{k+2}$ by definition, we have that $\left(f_{k+1}, f_{k+2}\right)$ is a generating pair of G for every integer k .

1.1.2 LEMMA. *Let $f(a, b)$ be a Fibonacci sequence on a group G generated by a and b . Then for any integer k the Fibonacci sequence $f\left(f_{k+1}, f_{k+2}\right)$ has the property that $f_n\left(f_{k+1}, f_{k+2}\right) = f_{n+k}$ for every integer n .*

Proof. Let ν be the epimorphism of F onto G with $x\nu = a$

and $y\nu = b$. Now $\varphi^k\nu$ is also an epimorphism of F onto G with

$$\begin{aligned} x\varphi^k\nu &= \left(x\varphi^k\right)\nu \\ &= w_{k+1}\nu \quad \text{by Lemma 1.1.1,} \\ &= f_{k+1} \quad \text{by definition,} \end{aligned}$$

and similarly $y\varphi^k\nu = f_{k+2}$.

It follows that

$$\begin{aligned} f_n\left(f_{k+1}, f_{k+2}\right) &= w_n\varphi^k\nu \quad \text{by definition,} \\ &= \left(x\varphi^{n-1}\right)\varphi^k\nu \text{ by Lemma 1.1.1,} \\ &= \left(x\varphi^{n+k-1}\right)\nu \\ &= w_{n+k}\nu \quad \text{by Lemma 1.1.1,} \\ &= f_{n+k} \quad \text{by definition.} \quad // \end{aligned}$$

1.1.3 LEMMA. *The subgroup* $\left\langle x^{-1}w_{k+1}, y^{-1}w_{k+2}\right\rangle^F$ *of* F *is invariant under the action of the automorphism* φ *of* F *, where* $x\varphi = y$ *and* $y\varphi = yx$.

Proof. By Lemma 1.1.1 we have $w_{l+1}\varphi = \left(x\varphi^l\right)\varphi = w_{l+2}$ for any integer l .

Now

$$\begin{aligned} \left(x^{-1}w_{k+1}\right)\varphi &= (x\varphi)^{-1}\left(w_{k+1}\varphi\right) \\ &= y^{-1}w_{k+2} , \end{aligned}$$

and

$$\left(y^{-1}w_{k+2}\right)\varphi = (y\varphi)^{-1}\left(w_{k+2}\varphi\right)$$
$$= x^{-1}y^{-1}w_{k+3}$$
$$= x^{-1}\left(y^{-1}w_{k+2}\right)xx^{-1}w_{k+1} ,$$

since $w_{k+3} = w_{k+2}w_{k+1}$, which proves the assertion. //

1.1.4 LEMMA. *Let $f(a, b)$ be a Fibonacci sequence on a group G generated by a and b. Then $f_{p+1} = f_1$ and $f_{p+2} = f_2$ for some integer p if and only if $f_{p+r+1} = f_{r+1}$ and $f_{p+r+2} = f_{r+2}$ for any integer r.*

Proof. Let ν be the epimorphism of F onto G with $x\nu = a$ and $y\nu = b$. We observe that for any integer r we have $f_{p+r+1} = f_{r+1}$ and $f_{p+r+2} = f_{r+2}$ if and only if $w_{r+1}^{-1}w_{p+r+1}$ and $w_{r+2}^{-1}w_{p+r+2}$ belong to the kernel of ν. But $w_{r+1}^{-1}w_{p+r+1}$ and $w_{r+2}^{-1}w_{p+r+2}$ belong to the kernel of ν if and only if $\left(w_{r+1}^{-1}w_{p+r+1}\right)\varphi^{-r} = w_1^{-1}w_{p+1}$ and $\left(w_{r+2}^{-1}w_{p+r+2}\right)\varphi^{-r} = w_2^{-1}w_{p+2}$ belong to the kernel of ν (using here the fact that for any integer k Lemma 1.1.1 implies that $w_k\varphi^{-r} = w_{k-r}$). This completes the proof. //

Let $f(a, b)$ be a Fibonacci sequence on a group G generated by a and b. Suppose there are integers p and r with

$$f_{p+r+1} = f_{r+1} , \; f_{p+r+2} = f_{r+2} . \tag{1.1}$$

By Lemma 1.1.4 this is true if and only if $f_{p+1} = f_1$ and $f_{p+2} = f_2$. In this case we say that the Fibonacci sequence f is periodic of period $|p|$. The least non-negative integer p for which (1.1) holds holds is called the *length* of the sequence. If $p = 0$ is the only integer p for which (1.1) holds then the Fibonacci sequence f is said to be aperiodic.

The next two lemmas deal with finite groups. The first is a remark due to B.H. Neumann.

1.1.5 LEMMA. *A two generator group is finite if and only if it has a finite number of generating pairs.*

Proof. Clearly a finite group will have a finite number of generating pairs.

Suppose now that G is a group with a finite number of generating pairs. We remark first that if (a, b) is a generating pair of G then both a and b have finite order. To see this we observe that for each integer r the pair $(a, a^r b)$ is a generating pair, and since these cannot all be distinct we must have $a^s = a^t$ for some different integers s and t. Similarly b has finite order. In particular G/G' is finite.

Now $G/Z(G)$ is finite since there can only be a finite number of distinct pairs (a^g, b^g) where g belongs to G. However a result of I. Schur* (see [21], 15.1.13) states that if $Z(G)$ is of finite index in G then G' is finite. We conclude now that G is finite. //

1.1.6 LEMMA. *Let $f(a, b)$ be a Fibonacci sequence on a finite group G generated by a and b. Then $f(a, b)$ is periodic.*

Proof. By Lemma 1.1.5 there are only finitely many generating pairs of G. Therefor some integers s and t with $s > t$ we have $(f_{s+1}, f_{s+2}) = (f_{t+1}, f_{t+2})$. By definition it follows that $f(a, b)$ is periodic with period $s - t$. //

REMARK. In §1.3 (see 1.3.6) we will show that the group

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* I wish to thank Dr L.G. Kovács for telling me of this result.

$\mathrm{gp}\{a, b;\ b^{-1}a^2b = a^{-2},\ a^{-1}b^2a = b^{-2}\}$ is an infinite group with a Fibonacci sequence $f(a, b)$ which has length 6 .

1.2 The freest group with a finite Fibonacci sequence

For each non-negative integer l we define G_l to be the group with presentation

$$\{x, y;\ x^{-1}w_{l+1},\ y^{-1}w_{l+2}\},$$

where w_{l+1} and w_{l+2} are the $(l+1)$-st and $(l+2)$-nd Fibonacci words.

Let (a, b) be a generating pair of G_l associated with this presentation and ν the epimorphsim of F onto G_l with $x\nu = a$ and $y\nu = b$. Then, since $x^{-1}w_{l+1}$ and $y^{-1}w_{l+2}$ belong to the kernel of ν , we have that $a = f_{l+1}(a, b)$ and $b = f_{l+2}(a, b)$. Consequently the sequence $f(a, b)$ is periodic of length l .

By definition $w_1 = x$ and $w_2 = y$ so clearly $G_0 = F$. The groups G_l for $l = 0, 1, \ldots, 7$ are determined in §1.3.

1.2.1 LEMMA. *The group* G_l *is the freest two generator group having a Fibonacci sequence* $f(a, b)$ *of length* l . *Any group* G *having a Fibonacci sequence* $f(c, d)$ *of period* l *is an epimorphic image of* G_l *by an epimorphism* ϑ *where* $a\vartheta = c$ *and* $b\vartheta = d$.

Proof. By our remarks above G_l has a Fibonacci sequence $f(a, b)$ of length l .

Let $f(c, d)$ be a Fibonacci sequence on a group generated by c and d and suppose that $f(c, d)$ has period l . Then $f_{l+1}(c, d) = f_1(c, d)$ and $f_{l+2}(c, d) = f_2(c, d)$. Let ν_1 be the epimorphism of F onto G_l with $x\nu_1 = a$, $y\nu_1 = b$, and let ν_2 be the epimorphism of F onto G with $x\nu_2 = c$, $y\nu_2 = d$. Then $x^{-1}w_{l+1}$ and $y^{-1}w_{l+2}$ belong to the kernel of ν_2 so that the kernel of ν_1 is contained in the kernel of ν_2 . It is clear now that there is an epimorphism ϑ with the required properties. //

1.2.2 LEMMA. *Let* (a, b) *be a generating pair of* G_l *associated with the presentation* $\left\{x, y; x^{-1}w_{l+1}, y^{-1}w_{l+2}\right\}$. *The automorphism* φ *of* F *defined by* $x\varphi = y$ *and* $y\varphi = yx$ *induces an automorphism* $\bar{\varphi}$ *of order* l *of* G_l *with* $a\bar{\varphi} = b$ *and* $b\bar{\varphi} = ba$.

Proof. By Lemma 1.1.3 the subgroup $\left\langle x^{-1}w_{l+1}, y^{-1}w_{l+2}\right\rangle^F$ of F is invariant under the action of φ . If ν is the epimorphism of F onto G_l with $x\nu = a$ and $y\nu = b$ then $\left\langle x^{-1}w_{l+1}, y^{-1}w_{l+2}\right\rangle^F$ equals the kernel of ν . Thus φ induces an automorphism $\bar{\varphi}$ of G_l defined by $g\bar{\varphi} = v\varphi\nu$ for any g in G_l , where $v\nu = g$. Consequently, for every integer k , using Lemma 1.1.1, we have $a\bar{\varphi}^k = x\varphi^k\nu = w_{k+1}\nu$ and $b\bar{\varphi}^k = y\varphi^k\nu = w_{k+2}\nu$. From the nature of the kernel of ν we see that $\bar{\varphi}$ has order exactly l . //

The following corollary should be compared with Satz 2(3) of [20].

1.2.3 COROLLARY. *If* l *is an odd positive integer and* (a, b) *is a generating pair of* G_l *associated with the presentation* $\left\{x, y; x^{-1}w_{l+1}, y^{-1}w_{l+2}\right\}$ *then* $[a, b]^2 = 1$.

Proof. By Lemma 1.2.2 there is an automorphism $\bar{\varphi}$ of G_l of order l with $a\bar{\varphi} = b$ and $b\bar{\varphi} = ba$.

But

$$[a, b]\bar{\varphi} = [b, ba] = b^{-1}a^{-1}b^{-1}bba = [b, a] .$$

Thus on the one hand as $\bar{\varphi}$ has order l we have $[a, b]\bar{\varphi}^{l} = [a, b]$ and on the other, since l is odd, we have $[a, b]\bar{\varphi}^{l} = [b, a]$. Thus $[a, b]^2 = 1$.

1.2.4 LEMMA. *Let l be an even non-negative integer and let (a, b) be a generating pair of G_l associated with the presentation $\left\{x, y; x^{-1}w_{l+1}, y^{-1}w_{l+2}\right\}$. The automorphism σ of F defined by $x\sigma = y$ and $y\sigma = x^{-1}$ induces an automorphism $\bar{\sigma}$ of order 4 of G_l with $a\bar{\sigma} = b$ and $b\bar{\sigma} = a^{-1}$.*

Proof. We show that $\left\langle x^{-1}w_{l+1}, y^{-1}w_{l+2}\right\rangle^F$ is invariant under σ .

An induction argument shows that for any non-negative integer r ,

$$w_{2r+1}\sigma = xw_{-2r+2}y^{-1}x^{-1}y$$

and

$$w_{2r+2}\sigma = xw_{-2r+1}^{-1}x^{-1} .$$

Consequently,

$$\left(x^{-1}w_{l+1}\right)\sigma = (x\sigma)^{-1}\left(w_{l+1}\sigma\right) = y^{-1}x\left(w_{-l+2}y^{-1}\right)x^{-1}y$$

and

$$\left(y^{-1}w_{l+2}\right)\sigma = (y\sigma)^{-1}\left(w_{l+2}\sigma\right) = x\left(xw_{-l+1}^{-1}\right)x^{-1} .$$

But

$$w_{-l+2}y^{-1} = (y\varphi^{-l})y^{-1} \quad \text{by Lemma 1.1.1,}$$

$$= \left(y(y\varphi^{l})^{-1}\right)\varphi^{-l} = \left(yw_{l+2}^{-1}\right)\varphi^{-l} ,$$

and similarly $xw_{-l+1}^{-1} = \left(w_{l+1}x^{-1}\right)\varphi^{-l}$. Now by Lemma 1.1.3, $\left\langle x^{-1}w_{l+1}, y^{-1}w_{l+2}\right\rangle^{F}$ is invariant under φ so that we conclude that it is invariant under σ .

Let ν be the epimorphism of F onto G_l with $x\nu = a$ and $y\nu = b$. Then it follows that σ induces an automorphism $\bar{\sigma}$ of G_l defined by $g\bar{\sigma} = v\sigma\nu$ for any g in G_l , where $v\nu = g$. //

1.3 The groups $G_0, G_1, \ldots, G_7$

In this section we determine $G_0, G_1, \ldots, G_7$. For l greater than 7 the groups G_l become harder to determine. We remark that apart from G_3 we have found no evidence to show that G_p is not cyclic when p is a prime number.

The following lemma will prove useful in describing the groups G_l .

1.3.1 LEMMA. *Let l be a non-negative integer. The group G_l has a presentation $\left\{x, y; w_{-r}^{-1}w_{l-r}, w_{-r+1}^{-1}w_{l-r+1}\right\}$ for any integer r .*

Proof. Let φ be the automorphism of Lemma 1.1.1. Then for any integers k and n we have $w_k\varphi^n = x\varphi^{k+n+1} = w_{k+n}$. Thus

$$\left(x^{-1}w_{l+1}\right)\varphi^{-r-1} = w_{-r}^{-1}w_{l-r} ,$$

and

$$\left(y^{-1}w_{l+2}\right)\varphi^{-r-1} = w_{-r+1}^{-1}w_{l-r+1}$$

for any integer r. The statement of the lemma is now clear. //

We may use Lemma 1.3.1 to find a presentation for G_l in which the relators have "minimal" length.

For example if $l = 7$ then

$$w_3^{-1}w_{-4} = (yx)^{-1}x^{-2}yx^{-2}yx^{-1}yx^{-2}yx^{-1}y ,$$

$$w_4^{-1}w_{-3} = (yxy)^{-1}y^{-1}xy^{-1}x^2y^{-1}x^2 ,$$

$$w_5^{-1}w_{-2} = \left(yxy^2x\right)^{-1}x^{-2}yx^{-1}y ,$$

$$w_6^{-1}w_{-1} = \left(yxy^2xyxy\right)^{-1}y^{-1}x^2 ,$$

$$w_7^{-1}w_0 = \left(yxy^2xyxy^2xy^2x\right)^{-1}x^{-1}y ,$$

so that we choose a presentation

$$\left(x, y;\ \left(yxy^2x\right)^{-1}x^{-2}yx^{-1}y,\ \left(yxy^2xyxy\right)^{-1}y^{-1}x^2\right)$$

for G_7.

As stated above G_0 is the free group of rank 2.

1.3.2 LEMMA. *G_1 is the trivial group.*

Proof. $G_1 = \mathrm{gp}(a, b;\ b = a,\ ba = b) = \mathrm{gp}(a, b;\ b = a = 1)$.

1.3.3 LEMMA. *G_2 is the trivial group.*

Proof. $G_2 = \mathrm{gp}(a, b;\ ba = a,\ bab = b) = \mathrm{gp}(a, b;\ b = a = 1)$.

1.3.4 LEMMA. *G_3 is the quaternion group of order* 8 .

Proof.

$$G_3 = \mathrm{gp}(a,\ b;\ ba = a^{-1}b,\ bab = a)$$

$$= \mathrm{gp}(a,\ b;\ aba = b,\ bab = a)$$

$$= \mathrm{gp}(a,\ b;\ b^2 = (ab)^2 = a^2)$$

which is a presentation of the quaternion group (see [5], p. 134).

1.3.5 LEMMA. *G_4 is the cyclic group of order* 5 .

Proof.

$$G_4 = \mathrm{gp}(a,\ b;\ ba = b^{-1}a^2,\ bab = a^{-1}b)$$

$$= \mathrm{gp}(a,\ b;\ b^2 = a,\ b = a^{-2})$$

$$= \mathrm{gp}(a,\ b;\ a^5 = 1,\ b = a^{-2})\ .$$

1.3.6 LEMMA. *G_5 is the cyclic group of order* 11 .

Proof.

$$G_5 = \mathrm{gp}(a,\ b;\ bab = b^{-1}a^2,\ bab^2a = a^{-1}b)\ .$$

We show that the relations $bab = b^{-1}a^2$ and $bab^2a = a^{-1}b$ are equivalent to the relations $b^{11} = 1$ and $a = b^{-8}$.

Suppose that $bab = b^{-1}a^2$ and $bab^2a = a^{-1}b$. By Corollary 1.2.3 we have that $[a,\ b]^2 = 1$. Consequently $bab^2a = a^{-1}b$ implies that

$$1 = ab^{-1}abab^2 = a^2a^{-1}b^{-1}abab^2 = a^2b^{-1}a^{-1}ba^2b^2\ .$$

However $bab = b^{-1}a^2$ implies that $a^2b^{-1}a^{-1} = b^2$ so that $1 = b^3a^2b^2$, or, $a^2 = b^{-5}$. Then from $bab = b^{-1}a^2$ we see that $a = b^{-8}$ and consequently $b^{11} = 1$.

Conversely,

$$baba^{-2}b = bb^{-8}bb^{16}b = b^{11} = 1$$

and

$$bab^2ab^{-1}a = bb^{-8}b^2b^{-8}b^{-1}b^{-8} = b^{-22} = 1 .$$

Thus $G_5 = \mathrm{gp}(a, b; b^{11} = 1, a = b^{-8})$.

1.3.7 LEMMA. *G_6 has the presentation $(x, y; x^2y^{-1}x^2y, y^2x^{-1}y^2x)$ and is an infinite metabelian group.*

Proof.

$$\begin{aligned} G_6 &= \mathrm{gp}(a, b; bab = a^{-2}ba^{-1}b, bab^2a = b^{-1}a^2) \\ &= \mathrm{gp}(a, b; b^{-1}a^{-2}b = a^2, ab^2a^{-1} = b^{-2}) \\ &= \mathrm{gp}(a, b; b^{-1}a^2b = a^{-2}, a^{-1}b^2a = b^{-2}) . \end{aligned}$$

Now $G_6/\langle a^2, b^2\rangle^{G_6}$ is the free product of two cyclic groups of order 2 so that G_6 is infinite.

Let H be the subgroup of G_6 generated by $[a, b], a^4, b^4$. We show that $H = G_6'$.

But $a^{-4} = [a^2, b]$ and $b^{-4} = [b^2, a]$ so H is contained in G_6' . Also $a^{-1}b^4a = b^{-4}$, $b^{-1}a^4b = a^{-4}$,

$[a, b]^a = a^{-2}b^{-1}aba = b^{-1}a^{-1}a^4ba = [b, a]a^{-4}$, and $[a, b]^b = [b, a]b^4$ so that H is normal in G_6' . Since $[a, b]$ belongs to H we conclude that $H = G_6'$.

Now

$$[a, b]^{a^4} = a^{-4}a^{-1}b^{-1}aba^4 = a^{-1}(a^{-4}b^{-1}a^{-4}aa^4ba^4) = a^{-1}b^{-1}ab ,$$

since $a^4ba^4 = b$, and similarly $[a, b]^{b^4} = [a, b]$. Also since $[a^2, b^2] = 1$ we have $[a^4, b^4] = 1$. Thus H is abelian which shows that G_6 is metabelian.

Remark. The dihedral groups with presentation $(x, y; x^n, y^2, xy^{-1}xy)$ for any positive integer n and the quaternion groups with presentation $\left(x, y; x^{2^\alpha}, y^4, y^{-2}x^{2^{\alpha-1}}, xy^{-1}xy\right)$ for any positive integer α are epimorphic images of G_6 .

1.3.8 Assertion. G_7 is the cyclic group of order 29 .

$$G_7 = \mathrm{gp}(a, b; bab^2a = a^{-2}ba^{-1}b, bab^2abab = b^{-1}a^2) .$$

A computer enumeration carried out by J. Cannon and confirmed by J. Watson establishes that G_7 is cyclic of order 29 . I know of no algebraic proof that G_7 is cyclic of order 29 .

It is not hard to show algebraically that $G_7' = G_7''$.

We first note that by Corollary 1.2.3 we have $[a, b]^2 = 1$ so that G_7'/G_7'' is an elementary abelian 2-group.

Putting $c = a^5b$ we obtain the relations $c^{a^2}c^{a^8}c^{-a^9}c^{-a^4}c^{-1} = 1$ and $c^{a^7}c^{a^{12}}c^{a^{16}}c^{a^{21}}c^{a^{25}}c^{a^{29}} = a^{29}$. From the first relation c belongs to G_7' .

Let c_i denote the coset $c^{a^i}G_7''$ for every integer i and e the coset G_7'' . It follows that the relations $c_9c_8c_4c_2c_0 = e$ and $c_i^2 = e$, for every integer i , hold in G_7'/G_7'' and there is an automorphism (induced by a) which is of order 29 such that $c_i^a = c_{i+1}$.

Now using the relation $c_9 = c_8c_4c_2c_0$ it is possible to express

$c_{29}(= c_0)$ in terms of $c_0, c_1, \ldots, c_8$ and in fact a calculation along these lines shows that $c_0 = c_8c_4c_3$. The relations $c_9 = c_8c_4c_2c_0$ and $c_0 = c_3c_4c_8$ can now be used to show that $c_0 = e$. Consequently G_7/G_7'' is cyclic of order 29, and since G_7/G_7' is also, we conclude that $G_7' = G_7''$.

1.4 The freest abelian group with a finite Fibonacci sequence. A problem of Netto

In this section a presentation of the group G_l/G_l' is studied. The presentation obtained is easy to express mainly because a Fibonacci sequence on an abelian group has a particularly simple form. The groups G_l/G_l' arise in an entirely different context which we discuss at the end of this section.

1.4.1 LEMMA. *Let $f(a, b)$ be a Fibonacci sequence on an abelian group G generated by a and b. Then $f_{n+1} = a^{u(n-1)}b^{u(n)}$ for each integer n, where $u(n-1)$ and $u(n)$ are the $(n-1)$-st and n-th Fibonacci numbers defined in §1.1.*

Proof. We prove the lemma by induction on n. When $n = 0$, as $u(-1) = 1$, $u(0) = 0$, and $f_1 = a$ by definition, the formula is true.

(a) Assume n is a positive integer and that the assertion is true for every non-negative integer less than n.

Then

$$f_{n+1} = f_n f_{n-1} \qquad \text{by definition,}$$

$$= a^{u(n-2)} b^{u(n-3)} a^{u(n-3)} b^{u(n-1)} \qquad \text{by induction hypothesis,}$$

$$= a^{u(n-2)+u(n-3)} b^{u(n-3)+u(n-1)} \qquad \text{since } a \text{ and } b \text{ commute,}$$

$$= a^{u(n-1)} b^{u(n-2)} ,$$

thus completing the induction step and the proof of the lemma for non-negative integers n .

(b) Assume n is a negative integer and that the assertion is true for every non-positive integer greater than n .

Then

$$f_{n+1} = f_{n+2}^{-1} f_{n+3} \qquad \text{by definition,}$$

$$= \left(a^{u(n)} b^{u(n+1)}\right)^{-1} a^{u(n+1)} b^{u(n+2)} \qquad \text{by induction hypothesis,}$$

$$= a^{u(n+1)-u(n)} b^{u(n+2)-u(n+1)} \qquad \text{since } a \text{ and } b \text{ commute,}$$

$$= a^{u(n-1)} b^{u(n-2)} ,$$

thus completing the induction step and the proof of the lemma for non-positive integers n . //

1.4.2 THEOREM. *Let* l *be a non-negative integer and let* $\delta = \left(u(l),\, u(l-1)-1\right)$ *be the greatest common divisor of* $u(l)$ *and* $u(l-1) - 1$. *Let* μ *and* ϑ *be integers such that* $\delta = \vartheta u(l) + \mu\left(u(l-1)-1\right)$. *Put* $\tau = \vartheta\left(1-u(l+1)\right) - \mu u(l)$ *and* $w = \left(v(l)-\left(1+(-1)^l\right)\right)/\delta$, *where* $v(l)$ *is the* l*-th Lucas number.*

Then, the relations $a^{u(l-1)-1} b^{u(l)} = 1$ *and* $a^{u(l)} b^{u(l+1)-1} = 1$ *are equivalent to the relation* $a^w = 1$, $b^w = 1$ *and* $a^\delta = b^\tau$.

Proof. Let

$$a^{u(l-1)-1} b^{u(l)} = 1 , \tag{1}$$

$$b^{u(l)} b^{u(l+1)-1} = 1 . \tag{2}$$

Using (1) and (2) we see that

$$a^{\delta} = a^{\vartheta u(l)+\mu(u(l-1)-1)} = b^{\vartheta(1-u(l+1))-\mu u(l)} = b^{\tau}$$

so that $a^{\delta} = b^{\tau}$. Also

$$\delta = (\vartheta-\mu)u(l) + \mu(u(l+1)-1)$$

so that using (1) and (2) we have

$$b^{\delta} = b^{(\vartheta-\mu)u(l)+\mu(u(l+1)-1)} = a^{-(\vartheta-\mu)(u(l-1)-1)-\mu u(l)} .$$

But

$$-(\vartheta-\mu)(u(l-1)-1) - \mu u(l) = -\vartheta(u(l+1)-1) + \vartheta u(l) - \mu(1-u(l-1)) - \mu u(l)$$
$$= \tau + \delta ,$$

so that $b^{\delta} = a^{\delta+\tau}$.

Clearly δ divides $u(l-1) - 1$ so that from (1) and since $a^{\delta} = b^{\tau}$ we have

$$b^{-u(l)} = a^{\delta(u(l-1)-1)/\delta} = b^{\tau((u(l-1)-1)/\delta)} ,$$

or, b has exponent $(\tau(u(l-1)-1)+\delta u(l))/\delta$. However,

$\tau(u(l-1)-1) + \delta u(l)$

$= \vartheta(u(l-1)-1)(1-u(l+1)) - \mu u(l)(u(l-1)-1) + \vartheta(u(l))^2 + \mu u(l)(u(l-1)-1)$

$= \vartheta(u(l))^2 - (1-u(l-1))(1-u(l+1))$.

But $(u(l))^2 - u(l-1)u(l+1) = (-1)^{l+1}$, (see 10.14.8 in [11]), so that

$(u(l))^2 - (1-u(l-1))(1-u(l+1))$

$$= (u(l))^2 - 1 + (u(l-1)+u(l+1)) - u(l-1)u(l+1)$$
$$= v(l) - (1+(-1)^{l}) \quad \text{since} \quad v(l) = u(l-1) + u(l+1) ,$$
$$= \delta w .$$

Thus $b^{\vartheta w} = 1$.

Similarly, as δ divides $u(l)$, and using (2), we see that

$b^{1-u(l+1)} = a^{\delta(u(l)/\delta)} = b^{\tau(u(l)/\delta)}$, or, b has exponent

$\left(\delta\left(1-u(l+1)\right)-\tau u(l)\right)/\delta$. But $\delta\left(1-u(l+1)\right) - \tau u(l) = \mu\left(v(l)-\left(1+(-1)^{l}\right)\right)$ so that $b^{\mu w} = 1$. Now $(\mu, \vartheta) = 1$ so that we conclude that $b^{w} = 1$. A similar argument to that above (using $b^{\delta} = a^{\delta+\tau}$) shows that $a^{w} = 1$.

Conversely, suppose that the relations $a^{w} = b^{w} = 1$, $a^{\delta} = b^{\tau}$ hold. Then $b^{\vartheta w} = 1$, and as $\vartheta w = \left(\tau\left(u(l-1)-1\right)+\delta u(l)\right)/\delta$ we see that $b^{-u(l)} = b^{\tau\left(u(l-1)-1\right)/\delta} = a^{\delta\left(u(l-1)-1\right)/\delta} = a^{u(l-1)-1}$ so that (1) holds. Similarly using $b^{\mu w} = 1$ we can show that (2) holds. This completes the proof. //

1.4.3 COROLLARY. *Let l be a non-negative integer and let δ, τ, w be as defined in Theorem* 1.4.2. *Then, under the canonical epimorphism of G_l onto G_l/G_l' the group G_l/G_l' has a presentation*

$$\left(x,\ y;\ x^{w},\ y^{w},\ x^{-\delta}y^{\tau},\ [x, y]\right) .$$

Proof. Under the canonical epimorphism of G_l onto G_l/G_l' we have that G_l/G_l' has a presentation $\left(x,\ y;\ x^{-1}w_{l+1},\ y^{-1}w_{l+2},\ [x, y]\right)$. Using Lemma 1.4.1 this is equivalent to the presentation $\left(x,\ y;\ x^{u(l-1)-1}y^{u(l)},\ x^{u(l)}y^{u(l+1)-1},\ [x, y]\right)$. The result now follows from Theorem 1.4.2. //

1.4.4 LEMMA. *Let w, τ, δ be defined as in Theorem* 1.4.2. *Then if (a, b) is a generating pair associated with the presentation of G_l/G_l' in Corollary* 1.4.3 *both a and b have order exactly w .*

Proof. We find an epimorphism of G_l/G_l' onto $\mathrm{gp}\left(c;\ c^{w} = 1\right)$. Firstly we have $\left(c^{\tau/\delta}\right)^{w} = \left(c^{w}\right)^{\tau/\delta} = 1$, $c^{w} = 1$, $\left(c^{\tau/\delta}\right)^{\delta} = c^{\tau}$ and $\left[c^{\tau/\delta},\ c\right] = 1$. Thus if η is the mapping of G_l/G_l' onto

$\mathrm{gp}(c;\ c^w = 1)$ defined by $a\eta = c^{\tau/\delta}$ and $b\eta = c$ then we see that η extends to an epimorphism. Thus b has order exactly w and a similar proof shows that a also has order exactly w . //

We state next, without proof, some useful information about the parameters in the presentation of G_l/G_l' in Theorem 1.4.2. The results can be obtained from those listed in 4.5 and 4.6 on p. 42 of A.-L. Arrhenius-Wold [1] (with a, d and v there replaced by l, δ and w).

1.4.5 LEMMA. *Let* l *be a non-negative integer and let* w, τ *and* δ *be defined as in Theorem* 1.4.2 .

(1) *If* l *is an integer of the form* $6n \pm 1$ *then* $\delta = 1$ *and* $w = v(l)$.

(2) *If* l *is an integer of the form* $6n - 3$ *then* $\delta = 2$ *and* $w = v(l)/2$.

(3) *If* l *is an integer of the form* $4n + 2$ *then* $\delta = v(l/2)$ *and* $w = v(l/2)$.

(4) *If* l *is an integer of the form* $4n$ *then* $\delta = u(l/2)$ *and* $w = 5u(l/2)$.

The groups G_l/G_l' have some relevance to a problem studied by A.-L. Arrhenius-Wold in [1]. This problem, attributed to E. Netto, asks for those positive integers m, n, r and s for which the group $\mathrm{gp}(a, b;\ a^m = b^n = 1,\ ba = a^r b^s)$ is finite and non-trivial. A.-L. Arrhenius-Wold considers the case $r = s = 2$. She proves that the relations $ba = a^2b^2$ and $ab^2a = ba^2b$ imply that a and b have the same order. Also she notes that if $m = n$ and n is either 4 or an odd positive integer the relation $ab^2a = ba^2b$ is a consequence of the relations $a^m = b^n = 1$ and $ba = a^2b^2$. The group which she then proceeds to study is

$$\mathrm{gp}(a,\ b;\ a^m = b^m = 1,\ ab^2a = ba^2b,\ ba = a^2b^2)\ ,$$

where m is a positive integer. It is shown to be finite and some information about the orders of various of its elements given. The methods used involve extensive combinatorial manipulations with elements in the group (particularly in the subgroup generated by ab and ba). In addition, permutation representations are constructed in order to prove the existence and non-triviality of the group.

We first elucidate briefly the structure of the group $K = \mathrm{gp}(a,\ b;\ ba = a^2b^2)$. Putting $d = ab$ and $c = b^{-1}$ it can be easily seen that $K = \mathrm{gp}\left(d,\ c;\ d^{c^2} = d^c d\right)$. When written in this form we see that K is one of the three groups described in Theorem 1 of E.S. Rapaport [19]. It is an extension of its derived group which is free of rank 2 by an infinite cyclic group. In fact K' is freely generated by d and d^c , and, if $f(d,\ d^c)$ is a Fibonacci sequence on K' then $d^{c^k} = f_{k+1}(d,\ d^c)$ for every integer k .

Let l be a positive integer and

$$G_l/G_l' = \mathrm{gp}(g,\ h;\ [g,\ h] = 1,\ g^{u(l-1)-1} = h^{-u(l)},\ g^{u(l)} = h^{1-u(l+1)})\ .$$

The automorphism $\bar{\varphi}$ of G_l , defined in Theorem 1.2.2, induces an automorphism φ' of order l of G_l/G_l' with $g\varphi' = h$ and $h\varphi' = hg$. We form the splitting extension of G_l/G_l' by φ' ;

$$\mathrm{gp}(g,\ h,\ c;\ [g,\ h] = 1,\ g^{u(l-1)-1} = h^{-u(l)},\ g^{u(l)} = h^{1-u(l+1)},\ c^l = 1,\ g^c = h,\ h^c = hg)$$

which defines the same group as

$$\mathrm{gp}(g, h, c; [g, h] = 1, c^l = 1, g^c = h, h^c = hg)$$

since the relations $g^{u(l-1)-1} = h^{-u(l)}$ and $g^{u(l)} = h^{1-u(l+1)}$ are a consequence of the other relations in the presentation. Now this last presentation defines the same group as

$$\mathrm{gp}\left(g, c; [g, g^c] = 1, c^l = 1, g^{c^2} = g^c g\right) .$$

Presenting the group with generators $a = c$ and $b = g^{-1}c^{-1}$ we arrive at the presentation $\mathrm{gp}(a, b; a^l = 1, ab^2a = ba^2b, ba = a^2b^2)$. Now remembering our comment above, that $b^l = 1$ is a relation in this group, it follows that this is precisely the group considered by A.-L. Arrhenius-Wold. It is the splitting extension of G_l/G_l' by its automorphism φ' of order l .

More generally, we remark that $K_l = \mathrm{gp}(a, b; a^l = 1, ba = a^2b^2)$ is the extension of G_l by its automorphism $\bar{\varphi}$ of order l . Since the relation $ba = a^2b^2$ implies $a^{-2} = b^2a^{-1}b^{-1} = bbaa^{-2}b^{-1} = ba^2b^2a^{-2}b^{-1}$ we see that a^{-2} and b^2 are conjugate. Thus, if l is even then $b^l = 1$ is a relation in K_l , while if l is odd in general $b^{2l} = 1$ is a relation in K_l but $b^l = 1$ is not.

When $l = 3, 4, 5$ and 7 , by our results in §1.3, we see that K_l is finite. However, when l is divisible by 6 (and there is an epimorphism of G_l onto G_6) the group K_l is an infinite group. A more detailed solution to Netto's problem requires a more detailed knowledge of the groups G_l .

1.5 Some factor groups of the groups G_l

For each non-negative integer l let H_l denote the group with presentation $\left(x, y; x^{-1}w_{l+1}, y^{-1}w_{l+2}, [x^2, y], [y^2, x]\right)$. In this section we study both the groups H_l and $G_l/[G_l', G_l]$.

When l is an odd positive integer H_l and $G_l/[G_l', G_l]$ define the same group which has been completely characterized by F. Roesler [20].

In §4 of [20] Roesler claims to know of no example of a finite non-abelian group with a Fibonacci sequence of odd length which is not nilpotent of class 2. We give such an example, due to R.M. Bryant, of an epimorphic image of G_{15}.

When l is an even positive integer H_l and $G_l/[G_l', G_l]$ do not define the same group. We study their respective presentations in Theorem 1.5.3 and Theorem 1.5.6.

We mention that Roesler defines a Fibonacci sequence based on the words w_n' defined in §1.1, but as was seen there the difference is only superficial. The following Lemma 1.5.1 is essentially a restatement of Hilfsatz 1 of [20].

1.5.1 LEMMA. *Let $f(a, b)$ be a Fibonacci sequence on* $\mathrm{gp}\left(a, b; [a^2, b] = [b^2, a] = 1\right)$.

If n is an odd positive integer then $f_n = b^{u(n-1)}a^{u(n-2)}$.

If n is an even positive integer then $f_n = b^{u(n-1)-1}a^{u(n-2)}b$.

Proof. The proof is by induction on n. When $n = 1$, as $u(-1) = 1$, $u(0) = 0$, and $f_1 = a$ by definition, we have

$f_1 = a = b^{u(0)}a^{u(-1)}$. Assume n is a positive integer and that the assertion is true for every positive integer less than n .

Then $f_n = f_{n-1}f_{n-2}$.

Thus if n is even $f_n = b^{u(n-2)}a^{u(n-3)}b^{u(n-3)-1}a^{u(n-2)}b$ by induction hypothesis, and as one of $u(n-3)$, $u(n-3) - 1$ is even we have $f_n = b^{u(n-1)-1}a^{u(n-2)}b$.

If n is odd $f_n = b^{u(n-2)-1}a^{u(n-3)}bb^{u(n-3)}a^{u(n-4)}$ by induction hypothesis, and as one of $u(n-3)$, $u(n-3) + 1$ is even we have $f_n = b^{u(n-1)}a^{u(n-2)}$ thus completing the induction step and the proof. //

We summarize (in our notations) some relevant results obtained by F. Roesler (see Satz 5 of [20]) in Theorem 1.5.2.

1.5.2 THEOREM. *Let l be an odd positive integer.*

(1) *If l is not divisible by* 3 *then $G_l/[G_l', G_l]$ is abelian.*

(2) *If l is divisible by* 3 *then $G_l/[G_l', G_l]$ is the direct product of the quaternion group of order* 8 *and a cyclic group. Moreover, in this case $G_l/[G_l', G_l]$ is a finite group of order* $2v(l)$ *and has a presentation given by*

$$(x,\ y;\ y^{v(l)},\ x^{-2}y^{v(l-1)-1},\ x^{-1}y^{-1}xyy^{v(l)/2}) .$$

Remark. The above presentation for $G_l/[G_l', G_l]$ is obtained under the canonical epimorphism of G_l onto $G_l/[G_l', G_l]$.

EXAMPLE. We now give an example, due to R.M. Bryant, which shows that there is a finite epimorphic image of G_{15} which is not nilpotent of class 2 .

For this purpose let V be an elementary abelian 2-group of rank 5 . There is an automorphism t of V with matrix

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{vmatrix} .$$

This matrix has the form from which the minimal polynomial is easily seen to be

$$\lambda^5 + \lambda^4 + \lambda^2 + \lambda + 1 = 0 ,$$

and it is not hard to check that t has order 31 .

Let $G = V\cdot\langle t\rangle$. Then, given any non-trivial element x in V the pair $\left(t^2, t^7x\right)$ is a generating pair of G . We form the Fibonacci sequence $f\left(t^2, t^7x\right)$. The members of the sequence may be listed (using the minimal polynomial above): t^2, t^7x, $t^9x^{t^2}$, $t^{16}x^{1+t^2+t^4}$, $t^{25}x^{t^3+t}$, $t^{10}x^{t^4+t^3+t+1}$, $t^4x^{t^2}$, $t^{14}x^{t^3+1}$, $t^{18}x^{t^4+1}$, tx^t, $t^{19}x^{t^4+t}$, $t^{20}x^{t^4+1}$, $t^8x^{t^{29}}$, $t^{28}x^{t^{28}}$, t^5x, t^2, t^7x, ... from which it becomes evident that the sequence has length 15 .

Moreover, $\left[t^2, t^7x\right] = x^{t^2}x$ which can be shown not to commute with t . It follows that G cannot be nilpotent of class 2 , although G is finite. Also by Lemma 1.2.1, G is an epimorphic image of G_{15} . //

1.5.3 LEMMA. *The sets of relators* $\left\{\left[x^2, y\right], \left[y^2, x\right], [x, y]^2\right\}$ *and* $\left\{\left[x, [x, y]\right], \left[y, [x, y]\right], [x, y]^2\right\}$ *are equivalent.*

Proof. It suffices to note that

$$[x, [x, y]] = [x, y]^{-x}[x, y]$$
$$= [x, y]^{-2x}[x, y]^{x}[x, y]$$
$$= [x, y]^{-2x}x^{-1}x^{-1}y^{-1}xyxx^{-1}y^{-1}xy$$
$$= [x, y]^{-2x}[x^2, y] ,$$

and similarly $[y, [x, y]] = [x, y]^{2y}[y^2, x]$. //

Remark. When l is an odd positive integer it follows from Corollary 1.2.3 that $[x, y]^2$ is a consequence of the relators $\{x^{-1}w_{l+1}, y^{-1}w_{l+2}\}$. Thus by Lemma 1.5.3 if l is odd $H_l = G_l/[G'_l, G_l]$.

If l is even and divisible by 3 the groups H_l and $G_l/[G'_l, G_l]$ differ. For example it will emerge that H_6 is infinite whereas $G_6/[G'_6, G_6]$ is finite.

1.5.3 THEOREM. *Let l be an even positive integer and let H_l be the group with presentation*

$$\{x, y ; x^{-1}w_{l+1}, y^{-1}w_{l+2}, [x^2, y], [y^2, x]\} .$$

(1) *If l is not divisible by 3 then H_l is abelian.*

(2) *If l is divisible by 3 then H_l is an extension of an abelian group by the quaternion group of order 8 and has a presentation*

$$\{x, y ; x^{w}, y^{w}, x^{\delta}y^{-\tau}, [x^2, y], [y^2, x]\} ,$$

where w, δ are defined in Theorem 1.4.2.

Proof. We remark that $1 = u(1) \equiv v(1)$, $1 = u(2) \equiv v(2)$, $u(3) \equiv v(3) \equiv 0$, $u(4) \equiv v(4) \equiv 1$ and $u(5) \equiv v(5) \equiv 1$ modulo 2 . It follows that $u(r)$ and $v(r)$ are even if and only if r is divisible by 3 .

Using Lemma 1.5.1 we see that

$$(x,\ y;\ x^{u(l-1)-1}y^{u(l)},\ y^{u(l+1)-1}x^{u(l)},\ [x^2,\ y],\ [y^2,\ x])$$

is a presentation for H_l . Let (a, b) be a generating pair associated with this presentation. If we suppose that l is not divisible by 3 we have shown that $u(l)$ is odd. But as $b^{u(l)} = a^{1-u(l-1)}$ we have $b^{u(l)}$ central in H_l . However as b^2 is also central we conclude that $[a, b] = 1$.

Suppose now that l is divisible by 3 .

Let (e, f) be a generating pair of G_3 , where G_3 by Lemma 1.3.4 is the quaternion group of order 8 . We remark that every generating pair of G_3 is associated with a presentation $(x,\ y;\ xyxy^{-1},\ yxyx^{-1})$. Moreover $[e^2, f] = [f^2, e] = 1$ so that $G_3 = H_3$.

By Lemma 1.2.1 there is a generating pair (c, d) of G_l and an epimorphism ψ of G_l onto $G_3 = H_3$ with $c\psi = e$ and $d\psi = f$. As $\langle [c^2, d], [d^2, c]\rangle^{G_l}$ is contained in the kernel of ψ we have that ψ induces an epimorphism $\bar{\psi}$ of H_l onto H_3 with $a\bar{\psi} = e$ and $b\bar{\psi} = f$.

Putting $u = abab^{-1}$ and $v = baba^{-1}$ we see that $\langle u, v\rangle^{H_l}$ is the kernel of $\bar{\psi}$.

Now $u^a = a^{-1}abab^{-1}a = bab^{-1}a = bab^{-2}ba = b^{-1}aba = u^b$ so that $u^a = u^b$ and similarly $v^a = v^b$. Also as $[u, b^2] = 1$ we have $u^{ab} = u^{b^2} = u$. It follows that $\langle u, v\rangle^{H_l}$ is generated by u, v, u^a and v^a .

We show that $\langle u, v\rangle^{H_l}$ is abelian.

Firstly $uv = abab^{-1}baba^{-1} = aba^2ba^{-1} = a^2b^2$, and $vu = baba^{-1}abab^{-1} = bab^2ab^{-1} = a^2b^2$, so that $uv = vu$ and consequently $u^av^a = v^au^a$.

Also $uv^a = abab^{-1}a^{-1}bab = abab^{-2}ba^{-2}abab = a^{-1}babb^{-1}aba = v^au$ so that $uv^a = v^au$ and consequently $u^av = u^av^{a^2} = v^{a^2}u^a = vu^a$.

Again $uu^a = abab^{-1}bab^{-1}a = aba^2b^{-1}a = a^4 = u^au$ and similarly $v^av = vv^a$. //

Remarks (1) If l is a positive integer and of the form $4n + 2$ then by Lemma 1.4.5 we have $\delta = w$. Now since by Lemma 1.4.4 both a and b have order exactly w in G_l/G_l' we must have τ divisible by w. Consequently in this case $(x, y; x^w, y^w, [x^2, y], [y^2, x])$ is a presentation for H_l.

(2) In most instances more details about the presentation of the subgroup $\langle u, v\rangle^{H_l}$ of H_l in Theorem 1.5.3 can be found. For example $H_6 = \mathrm{gp}(a, b;\ a^4 = b^4 = [a^2, b] = [b^2, a] = 1)$. A Reidemeister rewriting process (see Theorem 2.8 of [12]) can be used to show that $K = \mathrm{gp}(u, v;\ (uv)^2 = [u, v] = 1)$; the direct product of an infinite cyclic group and a cyclic group of order 2. //

We now proceed to the study of the groups $G_l/[G_l', G_l]$ with l an even positive integer. First, however, we need some preliminary results.

Let (a, b) be a generating pair of a group G which is nilpotent of class two. Then $[a, b]$ is central. Moreover every element of G has a representation as an element $b^m a^n [a, b]^s$ for some integers m, n and s. If $b^{m_1} a^{n_1} [a, b]^{s_1}$ and $b^{m_2} a^{n_2} [a, b]^{s_2}$ are two such elements then

$$b^{m_1} a^{n_1} [a, b]^{s_1} \cdot b^{m_2} a^{n_2} [a, b]^{s_2} = b^{m_1+m_2} a^{n_1+n_2} [a, b]^{n_1 m_2 + s_1 + s_2} . \quad (1.2)$$

1.5.4 LEMMA. *Let G be a two generator group which is nilpotent of class two. Let $f(a, b)$ be the Fibonacci sequence on G with (a, b) a generating pair of G. Put $g(1) = 0$ and define*

$g(t) = u(0)\{u(t-2)\}^2 + u(1)\{u(t-3)\}^2 + \ldots + u(t-2)\{u(0)\}^2$ *for any positive integer t greater than 1. Then*

$f_r = b^{u(r-1)} a^{u(r-2)} [a, b]^{g(r)}$ *for any positive integer r.*

Proof. Clearly the result is true for $r = 1$. Assume that r is a positive integer and that the result is true for every positive integer less than r.

Then

$$f_r = f_{r-1} f_{r-2}$$

$$= b^{u(r-2)} a^{u(r-3)} [a, b]^{g(r-1)} b^{u(r-3)} a^{u(r-4)} [a, b]^{g(r-2)}$$

by induction hypothesis,

$$= b^{u(r-1)} a^{u(r-2)} [a, b]^{\{u(r-3)\}^2 + g(r-1) + g(r-2)}$$ using (1.2).

Now

$$\begin{aligned}
&\{u(r-3)\}^2 + g(r-1) + g(r-2) \\
&= \{u(r-3)\}^2 + u(0)\{u(r-3)\}^2 + u(1)\{u(r-4)\}^2 + \ldots \\
&\quad + u(r-4)\{u(1)\}^2 + u(r-3)\{u(0)\}^2 + u(0)\{u(r-4)\}^2 + u(1)\{u(r-5)\}^2 + \ldots \\
&\qquad + u(r-5)\{u(1)\}^2 + u(r-4)\{u(0)\}^2 \\
&= \{u(r-3)\}^2 + u(1)\{u(r-4)\}^2 + \{u(1)+u(2)\}\{u(r-5)\}^2 + \ldots \\
&\qquad + \{u(r-4)+u(r-5)\}\{u(1)\}^2 + u(r-2)\{u(0)\}^2 \\
&= g(r)
\end{aligned}$$

which completes the induction step. //

1.5.5 LEMMA.

(1) $(u(1))^3 + (u(2))^3 + \ldots + (u(r-1))^3 =$

$$= 5^{-1}[2^{-1}(u(3r-1)-1)+3(1+(-1)^r u(r-2))] .$$

(2) $u(1)(u(2))^2 + u(2)(u(3))^2 + \ldots + u(r-2)(u(r-1))^2 =$

$$= 5^{-1}[2^{-1}(u(3r-2)-3)+(1+(-1)^{r-1}u(r-1))-2(1+(-1)^{r-1}u(r-3))] .$$

Proof. (1) This is just the formula proved in §11, p. 21, of Vorob'ev [22].

(2) Let $\alpha = 2^{-1}(1+\sqrt{5})$ and $\beta = 2^{-1}(1-\sqrt{5})$. We note that $\alpha\beta = -1$ and $u(n) = 5^{-1}(\alpha^n-\beta^n)$ for any integer n . Consequently, for any integer t we have

$$\begin{aligned} u(t)(u(t+1))^2 &= u(t)(5^{-1})(\alpha^{2t+2}+\beta^{2t+2}-2(-1)^{t+1}) \\ &= (5\sqrt{5})^{-1}[\alpha^{3t+2}-\beta^{3t+2}-(-1)^t(\alpha^{t+2}-\beta^{t+2})-2(-1)^{t+1}(\alpha^t-\beta^t)] \\ &= 5^{-1}[u(3t+2)-(-1)^t u(t+2)-2(-1)^{t+1}u(t)] . \end{aligned}$$

Now

$$\sum_{t=1}^{r-1} \alpha^{3t+2} = \alpha^2 \sum_{t=1}^{r-2} \alpha^{3t} = \alpha^2(\alpha^{3(r-1)}-\alpha^3)/(\alpha^3-1) .$$

Thus $\sum_{t=1}^{r-1} \alpha^{3t+2} = 2^{-1}(\alpha^{3r-2}-\alpha^4)$ as $\alpha^3 - 1 = 2\alpha$ and similarly

$\sum_{t=1}^{r-1} \beta^{3t+2} = 2^{-1}(\beta^{3r-2}-\beta^4)$ so that $\sum_{t=1}^{r-1} u(3t+2) = 2^{-1}(u(3r-2)-u(4))$.

Now

$$\sum_{t=1}^{r-2} (-1)^t u(t+2) = - \sum_{t=1}^{r-2} (-1)^{t+3} u(t+2)$$

$$= - \sum_{t=1}^{r} (-1)^{t+1} u(t) + \big(u(1)-u(2)\big)$$

$$= - \sum_{t=1}^{r} (-1)^{t+1} u(t)$$

$$= - \big(1+(-1)^{r-1} u(r-1)\big)$$

by (8) of §5, p. 8, of Vorob'ev [22]. Also

$\sum_{t=1}^{r-2} (-1)^{t+1} u(t) = \big(1+(-1)^{r-1} u(r-3)\big)$ by the same formula in [22].

Consequently

$$\sum_{t=1}^{r-2} u(t)\big(u(t+1)\big)^2 =$$

$$5^{-1}\big[2^{-1}\big(u(3r-2)-u(4)\big)+\big(1+(-1)^{r-1}u(r-1)\big)-2\big(1+(-1)^{r-1}u(r-3)\big)\big] . \qquad //$$

1.5.6 THEOREM. *Let l be an even positive integer divisible by neither 3 nor 5 . Then $G_l/[G_l', G_l]$ has a presentation*

$$\{x, y ; x^w, y^w, x^{-\tau}y^{\delta}, [x, y]^{\delta}, [x, [x, y]], [y, [x, y]]\} ,$$

where w, τ and δ are defined in Theorem 1.4.2.

Proof. The group $G_l/[G_l', G_l]$ has a presentation

$$\Big\{x, y ; x^{-1}w_{l+1}, y^{-1}w_{l+2}, [x, [x, y]], [y, [x, y]]\Big\} .$$

Let (a, b) be a generating pair associated with this presentation. We show that the presentation in the statement of the theorem is also associated with (a, b) .

From Lemma 1.5.4 we have $f_{l+1} = b^{u(l)}a^{u(l-1)}[a, b]^{g(l+1)}$ and $f_{l+2} = b^{u(l+1)}a^{u(l)}[a, b]^{g(l+2)}$. Since $[a, b]$ is central in $G_l/[G_l', G_l]$ and $f_{l+1} = a$, $f_{l+2} = b$ we have that

$$b^{u(l)}a^{u(l-1)-1}[a, b]^{g(l+1)} = 1$$

and

$$b^{u(l+1)-1}a^{u(l)}[a, b]^{g(l+2)} = 1 .$$

Now it follows from Corollary 1.4.3 that $b^{\delta}a^{\tau}$ belongs to the derived group which is contained in the centre. Thus b^{δ} is also central. Consequently we have $1 = [a, b^{\delta}] = [a, b]^{\delta}$.

Now $\delta = (u(l), u(l-1)-1) = (u(l), u(l+1)-1)$ so that $u(l) \equiv u(0) = 0$ modulo δ and $u(l+1) \equiv u(1) = 1$ modulo δ . Thus the sequence of Fibonacci numbers modulo δ is periodic and of length l . It follows that $u(l-r) \equiv u(-r)$ modulo δ for any integer r , and as $u(-r) = (-1)^{r+1}u(r)$ (see 10.14.5 of [11]) we have that $(u(l-r))^2 \equiv (u(r))^2$ modulo δ .

Hence

$$g(l+1) \equiv u(1)(u(2))^2 + u(2)(u(3))^2 + \dots + u(l-2)(u(l-1))^2$$

and

$$g(l+2) \equiv (u(1))^3 + (u(2))^3 + \dots + (u(l-1))^3 \text{ modulo } \delta .$$

But by Lemma 1.5.5 (1) we have

$$10g(l+2) = [(u(3l-1)-1)+6(1+(-1)^{l}u(l-2))] .$$

Thus $u(3l-1) \equiv u(-1) = 1$ modulo δ and $u(l-2) \equiv u(-2) = -1$ modulo δ so that $10g(l+2) \equiv 0$ modulo δ .

Again from Lemma 1.5.5 (2) we have

$$10g(l+1) = [(u(3l-2)-3)+2(1+(-1)^{l-1}u(l-1))-4(1+(-1)^{l-1}u(l-3))] .$$

But $u(3l-2) \equiv u(-2) = -1$ modulo δ , $u(l-1) \equiv u(-1) = 1$ modulo δ and $u(l-3) \equiv u(-3) = 2$ modulo δ so that $10g(l+1) \equiv 0$ modulo δ .

By assumption l is an even integer divisible by neither 3 nor 5 . Now considering the Fibonacci and Lucas sequences modulo 2

and modulo 5 it can be seen that: $v(r)$ and $u(r)$ are divisible by 2 if and only if r is divisible by 3, $v(r)$ is never divisible by 5, and 5 divides $u(r)$ if and only if 5 divides r. In view of Lemma 1.4.5 we conclude that neither 2 nor 5 divides δ. Thus $g(l+1) = g(l+2) \equiv 0$ modulo δ.

It follows now that $b^{u(l)}a^{u(l-1)-1} = 1$ and $b^{u(l+1)-1}a^{u(l)} = 1$ so that $a^{w} = b^{w} = 1$ and $a^{\delta} = b^{\tau}$ by Theorem 1.4.2.

Clearly the above argument is reversible. Thus we have (a, b) associated with both presentations. This proves the theorem. //

Remark. The restriction in Theorem 1.5.6 on l being divisible by neither 3 nor 5 is necessary for our argument.

It can be shown that

(1) $G_6/[G_6', G_6]$ has a presentation

$$\{x, y; x^4 = y^4, x^4 = [x, y]^2, [x, y]^4, [x, [x, y]], [y, [x, y]]\},$$

and

(2) $G_{12}/[G_{12}', G_{12}]$ has a presentation

$$\{x, y; x^{40} = y^{40}, x^{40} = [x, y]^4, [x, y]^8, [x, [x, y]], [y, [x, y]]\}.$$

1.6 Some groups with interesting Fibonacci sequence lengths

In general it is difficult to decide whether a Fibonacci sequence on a two generator group is periodic, and if so what length it has.

One observation is the following; if $f(a, b)$ is a Fibonacci sequence on a group G generated by a and b and ψ is an epimorphism of G onto a group H then the length of $f(a, b)$ is divisible by the length of the Fibonacci sequence $f(a\psi, b\psi)$ on H. In particular if $f(a\psi, b\psi)$ is aperiodic so is $f(a, b)$.

It is often useful to use the following method when finding the length of a Fibonacci sequence $f(a, b)$ on a group G. Let ν be the canonical epimorphism of F onto G with $x\nu = a$ and $y\nu = b$. We find the first integer r such that the automorphism φ^r of F (see Lemma 1.1.1) induces an automorphism of G; that is, there is an automorphism ϑ of G such that $(v\nu)\vartheta = v\varphi^r\nu$ for every v in F.

It follows that for every integer k we have

$$a\vartheta^k = (x\nu)\vartheta^k = x\varphi^{rk}\nu = f_{kr+1}$$

and $b\vartheta^k = f_{kr+1}$ so we see that the length of $f(a, b)$ is r times the order of ϑ. We use this method without comment in the sequel.

1.6.1 THEOREM. *The group* $\mathrm{gp}(a, b; a^3 = b^3 = (ab)^3 = 1)$ *has the aperiodic Fibonacci sequence* $f(a, b)$.

For any positive integer k *the group*

$\mathrm{gp}(c, d; c^3 = d^3 = (cd)^3 = (c^{-1}d)^{3k} = 1)$, *which is a finite group of order* $3k^2$, *has the Fibonacci sequence* $f(c, d)$ *of length* $8k$.

Proof. Let $f(a, b)$ be a Fibonacci sequence on

$G = \mathrm{gp}(a, b; a^3 = b^3 = (ab)^3 = 1)$. Then

$$f_1 = a, \quad f_2 = b, \quad f_3 = ba, \quad f_4 = bab, \quad f_5 = bab^2a,$$

$$f_6 = bab^2abab = baba^{-1} = a^{-1}b^{-1}a,$$

using $babab = a^{-1}$, $bab = a^{-1}b^{-1}a^{-1}$ and $a^{-2} = a$,

$$f_7 = a^{-1}b^{-1}abab^2a = a^{-1}b^{-1}b^{-1}a^{-1}ba = (a^{-1}b)^2a,$$

since $abab = b^{-1}a^{-1}$ and $b^{-2} = b$,

$$f_8 = (a^{-1}b)^2a\,a^{-1}b^{-1}a = a^{-1}b, \quad f_9 = (a^{-1}b)^3a \text{ and } f_{10} = (a^{-1}b)^3b.$$

We observe that

$$\begin{aligned}(a^{-1}b)^3a &= a^{-1}ba^{-1}ba^{-1}ba \\ &= a^{-1}b^{-1}ab\, b^{-1}a^{-1}bba^{-1}ba^{-1}\, ab\, b^{-1}a^{-1}ba \\ &= a^{-1}b^{-1}ab\, b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}\, b^{-1}a^{-1}ba \quad \text{since } b^2 = b^{-1}\ , \\ &= a^{[b,a]}\ , \text{ as } b^{-1}a^{-1}b^{-1}a^{-1}b^{-1} = a\ ,\end{aligned}$$

and similarly $(a^{-1}b)^3b = b^{[b,a]}$.

Now $[(a^{-1}b)^3, [b, a]] = 1$ since

$$\begin{aligned}b^{-1}a^{-1}baa^{-1}ba^{-1}ba^{-1}ba^{-1}b^{-1}ab &= b^{-1}a^{-1}b^{-1}a^{-1}ba^{-1}ba^{-1}b^{-1}ab \\ &= abba^{-1}ba^{-1}b^{-1}ab \quad \text{as } (ab)^3 = 1\ , \\ &= ab^{-1}a^{-1}ba^{-1}b^{-1}ab \quad \text{as } b^2 = b^{-1}\ , \\ &= ab^{-1}a^{-1}b^{-1}b^{-1}a^{-1}b^{-1}ab \quad \text{as } b = b^{-2}\ , \\ &= aabaabaab \quad \text{as } (ba)^3 = 1\ , \\ &= a^{-1}ba^{-1}ba^{-1}b \quad \text{as } a^2 = a^{-1}\ .\end{aligned}$$

In view of these comments we see that

$$f_{8k+1} = a^{[b,a]^k} = (a^{-1}b)^{3k}a \tag{1}$$

and

$$f_{8k+2} = b^{[b,a]^k} = (a^{-1}b)^{3k}b\ . \tag{2}$$

Consider now the epimorphic image of G given by $\mathrm{gp}(c, d;\ c^3 = d^3 = (cd)^3 = (c^{-1}d)^3 = 1)$. This is a non-abelian group of order 27 (see Table 1, p. 134 of [5]). By (1) and (2) above $f(a, b)$ is periodic with period 8, and since the abelian group $\mathrm{gp}(s, t;\ s^3 = t^3 = [s, t] = 1)$ has a Fibonacci sequence $f(s, t)$ of length 8, we conclude that $f(c, d)$ has length 8.

For each positive integer k let $(3, 3\,|\,3, 3k)$ denote

$gp(c, d; c^3 = d^3 = (cd)^3 = (c^{-1}d)^{3k} = 1)$ and the Fibonacci sequence $f(c, d)$ on $(3, 3|3, 3k)$ by $f(c, d)(k)$. Then we have shown that $f(c, d)(1)$ has length 8. Since $(3, 3|3, 3)$ is an epimorphic image of $(3, 3|3, 3k)$ for each k we have that $f(c, d)(k)$ has length $8k'$ for some k'. Now from (1) and (2) and the fact that $(3, 3|3, 3k)$ has order $8k^2$ (see [4]) we conclude that $f(c, d)(k)$ has length exactly $8k$. It also follows that the Fibonacci sequence $f(a, b)$ on G is aperiodic. //

1.6.2 LEMMA. *The group* $G = gp(a, b; a^4 = b^2 = (ab)^4 = 1)$ *is metabelian.*

Proof. Let H be the subgroup of G generated by $[a, b]$ and $[a, b]^a$. Then H is a subgroup of G'. We show that H is normal in G from which it will follow that $H = G'$.

But

$$[a, b]^{a^2} = a^{-2}a^{-1}b^{-1}aba^2 = ababaa = ba^{-1}ba = [b, a] ,$$

$$[a, b]^b = b^{-1}a^{-1}b^{-1}abb = b^{-1}a^{-1}ba = [b, a]$$

and

$$[a, b]^{ab} = b^{-1}a^{-1}a^{-1}b^{-1}abab = baababab = bab^{-1}a^{-1}$$

$$= a^{-1}ba^{-1}ba^{-1}a^{-1} = a^{-1}b^{-1}a^{-1}baa = [b, a]^{a} .$$

We show next that H is abelian and this will imply that G is metabelian. Now

$$b^{-1}a^{-1}baa^{-1}a^{-1}b^{-1}abaa^{-1}b^{-1}ab = ba^{-1}ba^{-1}ba^2b$$

$$= ba^{-1}ba^{-1}ba^{-1}a^{-1}b = ab^{-1}a^{-1}b = a^{-1}(a^2b^{-1}a^{-1}ba^{-1})a = a^{-1}(a^{-1}b^{-1}ab)a$$

so that $[[a, b], [a, b]^a] = 1$. //

1.6.3 LEMMA. *The element* $[a, b]^2$ *is central in the group*

$$\mathrm{gp}\left(a,\ b;\ a^4 = b^4 = (ab)^4 = \left(a^{-1}b\right)^4 = \left(a^2b\right)^4 = \left(b^2a\right)^4 = 1\right) .$$

Proof.

$$\begin{aligned}
ba^{-1}b^{-1}aba^{-1}b^{-1}abb^{-1} &= ba^{-1}b^{-1}aba^{-1}b^{-1}a \\
&= ba^{-1}b^{-1}aba^{-1}a^{-1}ba^{-1}ba^{-1}b \quad \text{as} \quad \left(a^{-1}b\right)^4 = 1 , \\
&= ba^{-1}b^{-1}aa^2b^{-1}a^2b^{-1}aba^{-1}b \quad \text{as} \quad \left(a^{-2}b\right)^4 = 1 , \\
&= ba^{-1}b^{-1}a^{-1}b^{-1}a^2b^{-1}aba^{-1}b \quad \text{as} \quad a^4 = 1 , \\
&= bbabaa^2b^{-1}aba^{-1}b \quad \text{as} \quad (ab)^4 = 1 , \\
&= b^2aba^{-1}b^{-1}aba^{-1}b \quad \text{as} \quad a^4 = 1 , \\
&= b^2abbababaaba^{-1}b \quad \text{as} \quad (ab)^4 = 1 , \\
&= b^2ab^2ababa^2ba^{-1}b \\
&= a^{-1}b^{-2}a^{-1}b^{-2}baba^2ba^{-1}b \quad \text{as} \quad \left(b^2a\right)^4 = 1 , \\
&= a^{-1}b^{-2}a^{-1}b^{-1}a^{-1}b^{-1}a^{-2}b^{-1}a^{-2}a^{-1}b \quad \text{as} \quad \left(a^2b\right)^4 = 1 , \\
&= a^{-1}b^{-2}a^{-1}b^{-1}a^{-1}b^{-1}a^{-2}b^{-1}ab \quad \text{as} \quad a^4 = 1 , \\
&= a^{-1}b^{-1}aba^{-1}b^{-1}ab \quad \text{as} \quad (ab)^4 = 1 .
\end{aligned}$$

Thus $\left[[a,\ b]^2,\ b\right] = 1$. Now there is an automorphism of the group interchanging a and b so that $\left[[a,\ b]^2,\ a\right] = 1$ as well. //

1.6.4 THEOREM. *The group* $\mathrm{gp}\left(a,\ b;\ a^4 = b^2 = (ab)^4 = 1\right)$ *has the aperiodic Fibonacci sequence* $f(a,\ b)$.

For any positive integer k *the group* $\mathrm{gp}\left(c,\ d;\ c^4 = d^2 = (cd)^4 = \left(c^2d\right)^{2k} = 1\right)$ *, which is of order* $16k^2$ *, has the Fibonacci sequence* $f(c,\ d)$ *of length* $6k$.

The group $\mathrm{gp}\left(g,\ h;\ g^4 = h^4 = (gh)^4 = \left(g^{-1}h\right)^4 = \left(g^2h\right)^4 = \left(gh^2\right)^4 = 1\right)$ *has the Fibonacci sequence* $f(g,\ h)$ *of length* 12. *The two generator Burnside group of exponent* 4 *has a Fibonacci sequence of* *length* 12 .

Proof. Let $f(s, t)$ be a Fibonacci sequence on

$$G = \mathrm{gp}\left(s, t; t^4 = \left(st^2\right)^4 = \left(st^3\right)^4 = 1\right) .$$

Then

$$\begin{aligned} f_6^{[t,s]} &= s^{-1}t^{-1}sttst^2ststt^{-1}s^{-1}ts \\ &= s^{-1}t^{-1}st^2st^2st^2s \\ &= s^{-1}t^{-3} \quad \text{as} \quad \left(st^2\right)^4 = 1 , \\ &= s^{-1} \quad \text{as} \quad t^4 = 1 . \end{aligned}$$

Also

$$\begin{aligned} f_7^{[t,s]} &= s^{-1}t^{-1}sttst^2stst^2st^2st^{-1}s^{-1}ts \\ &= s^{-1}t^{-1}st^2st^2stst^2st^2st^{-1}s^{-1}ts \\ &= s^{-1}t^{-1}t^{-2}s^{-1}t^{-1}t^{-2}s^{-1}t^{-2}t^{-1}s^{-1}ts \quad \text{as} \quad \left(t^2s\right)^4 = 1 , \\ &= s^{-1}t^{-3}s^{-1}t^{-3}s^{-1}t^{-3}s^{-1}ts \\ &= t^4s \quad \text{as} \quad \left(t^3s\right)^4 = 1 , \\ &= s \quad \text{as} \quad t^4 = 1 . \end{aligned}$$

It follows that $f_7 = s^{[s,t]}$ and $f_8 = t^{[s,t]}$ so that if k is any positive integer,

$$f_{6k+1} = s^{[s,t]^k} \tag{i}$$

and

$$f_{6k+2} = t^{[s,t]^k} . \tag{ii}$$

(1) The group

$$H = \mathrm{gp}\left(g, h; g^4 = h^4 = (gh)^4 = \left(g^{-1}h\right)^4 = \left(g^2h\right)^4 = \left(gh^2\right)^4 = 1\right)$$

is an epimorphic image of G in such a way that by (i) and (ii) if

$f(g, h)$ is a Fibonacci sequence on H then $f_{13} = g^{[g,h]^2}$ and $f_{14} = h^{[g,h]^2}$. From Lemma 1.6.3 we have that $[g, h]^2$ is central in H so that $f_{13} = g$ and $f_{14} = h$. Thus $f(g, h)$ is periodic of period 12.

Now B, the two generator Burnside group of exponent 4, is an epimorphic image of H and so also has a periodic Fibonacci sequence of period 12. If B has period 6 then B is an epimorphic image of G_6. But in Lemma 1.3.6 it was shown that G_6' is abelian and generated by at most 3 elements. This implies that B has order at most $4^3 4^2 = 2^{10}$, whereas B has order 2^{12} (see p. 80 of [5]). Thus B, and consequently H, cannot have a Fibonacci sequence of length 6. Since lengths 2, 3 and 4 are easily discounted we conclude that H has the Fibonacci sequence $f(g, h)$ of length 12 and that B also has a Fibonacci sequence of length 12.

(2) The group

$$K(k) = \mathrm{gp}\{c, d;\ c^4 = d^2 = (cd)^4 = (c^2d)^{2k} = 1\}$$

is an epimorphic image of G for each positive integer k in such a way that if $f(c, d) = f(c, d)(k)$ is a Fibonacci sequence on $K(k)$ then $f_{6k+1} = c^{[c,d]}$ and $f_{6k+2} = d^{[c,d]}$ from (i) and (ii).

We remark that

$$c^{[c,d]} = d^{-1}c^{-1}dccc^{-1}d^{-1}cd = d^{-1}c^{-1}dcd^{-1}cd$$

$$= dc^2cdcdcd = dc^2dc^2c = (dc^2)^2c\ ,$$

and

$$d^{[c,d]} = d^{-1}c^{-1}dcdc^{-1}d^{-1}cd = dc^{-1}d^{-1}cdc^{-1}d^{-1}cd = d[c, d]^2\ .$$

In view of Lemma 1.6.2 we have that K is metabelian. We note that $(dc^2)^2 = d^{-1}c^{-2}dc^2$ belongs to K'. Thus it follows that $c^{[c,d]^k} = (dc^2)^{2k}c$ and $d^{[c,d]^k} = d[c, d]^{2k}$ so that $f_{6k+1} = c$ and $f_{6k+2} = d$ if and only if $(dc^2)^{2k} = [c, d]^{2k} = 1$. But

$$[c, d]^2 = c^{-1}d^{-1}cdc^{-1}d^{-1}cd = c^{-1}c^{-1}dc^{-1}dc^{-1}c^{-1}d^{-1}cd$$
$$= c^{-2}dc^{-2}cdc^{-2}dcd = c^2dc^2ddc^{-1}(c^2dc^2d)cd = (c^2d)^2(c^2d)^{2cd}$$

so that $[c, d]^{2k} = (c^2d)^{2k}((c^2d)^{2k})^{cd}$.

Consequently $[c, d]^{2k} = 1$ if we have $(dc^2)^{2k} = 1$. We see then that $f_{6k+1} = c$ and $f_{6k+2} = d$ if and only if $(c^2d)^{2k} = 1$.

It follows now that $f(c, d)(k)$ has period $6k$. But the abelian group $\text{gp}(v, w; v^4 = w^2 = [v, w] = 1)$ has a Fibonacci sequence $f(v, w)$ of length 6 and is an epimorphic image of $K(k)$ for each positive integer k. Thus $f(c, d)(k)$ has length $6k'$ for some positive integer k'. We conclude that since $K(k)$ has order $16k^2$ (by [4]), $k' = k$ and $f(c, d)(k)$ has length $6k$. Also, the group $\text{gp}(a, b; a^4 = b^2 = (ab)^4 = 1)$ must have an aperiodic Fibonacci sequence $f(a, b)$. //

We conclude this chapter by giving some more examples of groups with Fibonacci sequences of small length. Since the techniques used are similar to those used above, and do not show anything new, we omit proofs.

(i) The group $\text{gp}(a, b; b^3 = (a^{-1}b)^4 = (a^{-2}b)^4 = (a^{-3}b)^3 = 1)$ has a Fibonacci sequence $f(a, b)$ of length 10. It has as an epimorphic image the group $\text{gp}(c, d; c^3 = d^3 = (cd)^4 = (c^{-1}d)^4 = 1)$

which has order 168 (see Table 6, p. 138 of [5]; it is the group $PSL(2, 7)$).

(ii) The infinite group $gp(a, b;\ a^4 = (b^2a)^2 = 1,\ a^2b = ba^2)$ has a Fibonacci sequence $f(a, b)$ of length 12 .

(iii) The infinite group $gp(a, b;\ a^4 = b^4 = (a^2b)^2 = (b^2a)^2 = 1)$ has a Fibonacci sequence $f(a, b)$ of length 18 .

CHAPTER II

T-SYSTEMS OF GROUPS

2.1 Preliminaries

Let G be an n-generator group. A generating n-tuple of G,

$$\mathrm{g} = (g_1, g_2, \ldots, g_n)$$

is an ordered set of n elements which generate G. Let $\Gamma(G)$ denote the set of generating n-tuples of G.

Let F denote the free group of rank n with generating n-tuple $\mathrm{x} = (x_1, x_2, \ldots, x_n)$, and A its automorphism group.

With each α in A such that

$$x_i\alpha = w_i(x_1, x_2, \ldots, x_n)$$

for $i = 1, 2, \ldots, n$, and $\mathrm{g} = (g_1, g_2, \ldots, g_n)$ in $\Gamma(G)$ put

$$\mathrm{g}\alpha = (g_1', g_2', \ldots, g_n'),$$

where $g_i' = w_i(g_1, g_2, \ldots, g_n)$ for $i = 1, 2, \ldots, n$.

It is clear that the elements of A act as permutations of $\Gamma(G)$. Moreover, if β and γ are elements of A then $\mathrm{g}(\gamma\beta) = (\mathrm{g}\beta)\gamma$, so that it follows that A has an antirepresentation as a group of permutations of $\Gamma(G)$.

When $G = F$ the permutations of $\Gamma(G)$ induced by A are Nielsen transformations of rank n (see p. 130 and p. 162 of [12]).

Let ψ be an epimorphism of a group G onto a group H and let $\mathrm{g} = (g_1, g_2, \ldots, g_n)$ belong to $\Gamma(G)$. We set

$$\mathrm{g}\psi = (g_1\psi, g_2\psi, \ldots, g_n\psi),$$

and note that $\mathrm{g}\psi$ belongs to $\Gamma(H)$.

In particular if B denotes the automorphism group of G and ψ belongs to B then $g\psi$ belongs to $\Gamma(G)$. It follows that B has a representation as a group of permutations of $\Gamma(G)$.

2.1.1 LEMMA. *Let* ψ *be an epimorphism of a group* G *onto a group* H. *If* g *belongs to* $\Gamma(G)$ *and* α *is an element of* A *then* $(g\alpha)\psi = (g\psi)\alpha$.

Proof. Suppose that $x_i\alpha = w_i\{x_1, x_2, \ldots, x_n\}$ for $i = 1, 2, \ldots, n$ and let $g = \{g_1, g_2, \ldots, g_n\}$. We write $w_i(g)$ as an abbreviation for $w_i\{g_1, g_2, \ldots, g_n\}$. Then,

$$\begin{aligned}(g\psi)\alpha &= \{w_1(g\psi), \ldots, w_n(g\psi)\}\\ &= \{w_1(g)\psi, \ldots, w_n(g)\psi\}\\ &= \{w_1(g), \ldots, w_n(g)\}\psi\\ &= (g\alpha)\psi . \quad //\end{aligned}$$

Let $\overline{A}$ denote the group of permutations of $\Gamma(G)$ induced by A and let $\overline{B}$ denote the group of permutations of $\Gamma(G)$ induced by B. By Lemma 2.1.1 the elements of $\overline{A}$ centralize the elements of $\overline{B}$. Therefore $\overline{A}\overline{B}$ is a group; the group generated by the permutations of $\Gamma(G)$ induced by A and B.

The transitivity sets gA of $\Gamma(G)$ under the action of elements of A are called *A-classes*.

The transitivity sets gB of $\Gamma(G)$ under the action of elements of B are called *B-classes*.

The transitivity sets gAB of $\Gamma(G)$ under the action of elements of A and of B are called *T-systems*.

T-systems and the related concepts as presented here were defined by B.H. Neumann and Hanna Neumann in [14].

We note that if h belongs to gAB then $h = g\alpha\beta$ for some α

in A and β in B since by Lemma 2.1.1 the elements of $\bar{A}$ centralize the elements of $\bar{B}$.

2.2 A-classes of generating pairs

Let K denote the free abelian group of rank 2 and let $k = (k_1, k_2)$ be a generating pair of K. Let M denote the automorphism group of K. Then M can be described as the group of all δ with $k_i\delta = k_1^{s_i}k_2^{t_i}$ for $i = 1, 2$ where s_1, s_2, t_1 and t_2 are integers satisfying $s_1t_2 - s_2t_1 = \pm 1$.

We define a mapping f of A into M. Thus let α be an element of A such that $x_i\alpha = w_i(x_1, x_2)$ for $i = 1, 2$ and let $\sigma_i(w_j)$ denote the exponent sum of w_j on x_i for $i = 1, 2$ and $j = 1, 2$. It can be shown that τ defined by $k_i\tau = k_1^{\sigma_1(w_i)}k_2^{\sigma_2(w_i)}$ for $i = 1, 2$ belongs to M. We put $\alpha f = \tau$. On p. 168 of [12] it is shown that f defines an epimorphism of A onto M.

The kernel of f is described in the following theorem of J. Nielsen (see Corollary N.4, p. 169 of [12]).

2.2.1 THEOREM. *The kernel of the epimorphism f of the automorphism group of the free group of rank 2 onto the automorphism group of the free abelian group of rank 2 consists of all inner automorphisms of the free group of rank 2.*

2.2.2 LEMMA. *Let H be a cyclic group generated by h and let α be an element of A such that $(1, h)\alpha = (1, h^{\varepsilon})$ where $\varepsilon = \pm 1$. Then, $\alpha = \bar{\alpha}\gamma$ where γ is an inner automorphism of F and $\bar{\alpha}$ is the element of A defined by $x_1\bar{\alpha} = x_1^{\varepsilon'}$ and $x_2\bar{\alpha} = x_1^{s}x_2^{\varepsilon}$,* *with* $\varepsilon' = \pm 1$ *and* s *some integer.*

Proof. Let f denote the epimorphism of A onto M defined above and let α belong to A with $x_i\alpha = w_i(x_1, x_2)$ for $i = 1, 2$. Then clearly $(1, h)\alpha = (w_1(1, h), w_2(1, h)) = (h^{t_1}, h^{t_2})$ where $t_i = \sigma_2(w_i)$ for $i = 1, 2$. Put $s_i = \sigma_1(w_i)$ for $i = 1, 2$.

But as $(1, h)\alpha = (1, h^{\varepsilon})$ we have $t_1 = 0$ and $t_2 = \varepsilon$. From $s_1t_2 - s_2t_1 = \pm 1$ we conclude that $s_1 = \varepsilon'$ where $\varepsilon' = \pm 1$. However, there is an element $\bar{\alpha}$ in A defined by $x_1\bar{\alpha} = x_1^{\varepsilon'}$ and $x_2\bar{\alpha} = x_1^s x_2^{\varepsilon}$ and clearly $\bar{\alpha}f = \alpha f$. The result now follows from Theorem 2.2.1. //

The following lemma may be obtained from Satz 5.2 of [14].

2.2.3 LEMMA. *Let* I *be the subgroup of* A *consisting of all inner automorphisms of the free group of rank* 2. *If* $\bar{I}$ *denotes the group of permutations of* $\Gamma(G)$ *induced by* I *then* $\bar{I}$ *is contained in the group of permutations of* $\Gamma(G)$ *induced by the inner automorphisms of* G.

We now make some elementary observations about A-classes of generating pairs.

2.2.4 LEMMA. *A cyclic group has one* A*-class of generating pairs.*

Proof. Let b be a generator of a cyclic group and suppose that (b^i, b^j) is a generating pair for some integers i and j.

Let $d = (i, j) = ip + jq$ for some integers p and q with $(p, q) = 1$. There are integers s_2, t_2 such that $s_2p - t_2q = 1$ so that there is an α in A with

$$\left(b^i, b^j\right)\alpha = \left(b^{pi+qj}, b^{s_2i+t_2j}\right) = \left(b^d, b^{s_2i+t_2j}\right) .$$

Further since $d = (i, j)$ there is an integer r with $s_2 i + t_2 j = rd$.

It follows that $\left(b^i, b^j\right)$ is in the same A-class as

$\left(b^d, b^{-rd}b^{s_2i+t_2j}\right) = \left(b^d, b^0\right)$.

Now b^d generates the cyclic group so there is an integer n with $b^{nd} = b$. Thus $\left(b^i, b^j\right)$ is in the same A-class as the pairs $\left(b^d, b^{nd}\right) = \left(b^d, b\right)$ and $\left(b^d b^{-d}, b\right) = (1, b)$. This completes the proof. //

It is well known ([15]) that every finitely generated abelian group has one T-system. However a finitely generated abelian group may have more than one A-class. The A-classes of generating n-tuples of n-generator abelian groups have been determined in Theorem 4.2 of [7].

2.2.5 LEMMA. *Let G be a two generator group and N a normal subgroup of G such that G/N is cyclic. Then G has a generating pair (a, b) with a belonging to N and bN a generator of G/N. Also, any generating pair of G is in the same A-class as a generating pair (c, bd) where c and d belong to N.*

Proof. Let ν denote the natural epimorphism of G onto G/N. Let $\left(g_1, g_2\right)$ be a generating pair of G. Then $\left(g_1\nu, g_2\nu\right)$ is a generating pair of G/N so that by Lemma 2.2.4 there is an α in A such that $\left(g_1\nu, g_2\nu\right)\alpha = (1, gN)$, where gN is a generator of G/N. Now using Lemma 2.2.1 we see that

$$\left(\left(g_1, g_2\right)\alpha\right)\nu = \left(\left(g_1, g_2\right)\nu\right)\alpha = (1, gN) ,$$

so that $\left(g_1, g_2\right)\alpha = (a, b)$ where $bN = gN$ and a belongs to N.

Since the generating pair (g_1, g_2) was arbitrary the last part of the lemma also follows. //

The following theorem of J. Nielsen states a characteristic property of elements in A (see N.3.9, p. 165 of [12]).

2.2.6 THEOREM. *Let* α *be an element of the automorphism group of the free group of rank* 2 *generated by* x *and* y. *Then*

$$[x, y]\alpha = [x, y]^{\varepsilon w},$$

where w *is a word in* x *and* y *and* $\varepsilon = \pm 1$.

2.3 *T*-systems and presentations

Let G be a group and let g belong to $\Gamma(G)$. Then there is a unique epimorphism θ of F onto G with $\mathsf{x}\theta = \mathsf{g}$. Let the kernel of θ be denoted by $R(\mathsf{g})$.

If β is an automorphism of G then $R(\mathsf{g}) = R(\mathsf{g}\beta)$ so that g and $\mathsf{g}\beta$ are associated with the same presentations of G.

When α is an automorphism of the free group of rank n then it is shown on p. 114 of [14] that $R(\mathsf{g}\alpha) = R(\mathsf{g})\alpha^{-1}$. Thus if $(x_1, x_2, \ldots, x_n; r, s, t, \ldots)$ is a presentation of G associated with g then $\left(x_1, x_2, \ldots, x_n; r\alpha^{-1}, s\alpha^{-1}, t\alpha^{-1}, \ldots\right)$ is a presentation of G associated with $\mathsf{g}\alpha$.

T-systems were introduced in [14] in connection with the study of certain characteristic subgroups of free groups. In fact

$$\bigcap_{\mathsf{h} \in \mathsf{g}AB} R(\mathsf{h})$$ is the unique maximal characteristic subgroup of F contained in $R(\mathsf{g})$ (see Satz 7.1 of [14]). If G has one T-system then this subgroup is not only characteristic in F but "hypercharacteristic" (see Satz 7.2 of [14]).

However, the importance of T-systems lies also in their use as a means of classifying presentations of a group. Clearly important invariants of a presentation like the minimal number of relators for a presentation are invariants of presentations associated with generating tuples in the same T-system. On the other hand according to [2] G. Higman has shown that the group $\mathrm{gp}(a, b; b^{-1}a^2b = a^3)$ has a presentation associated with the generating pair (a^4, b) which requires at least two relators. Therefore the pairs (a, b) and (a^4, b) lie in different T-systems. In §2.6 we study in some detail T-systems and presentations for the group $\mathrm{gp}(a, b; b^{-1}a^2b = a^3)$.

2.4 Magnus's conjecture

According to J. McCool and A. Pietrowski in [13], W. Magnus has conjectured that if G is a one relator group, $(x_1, x_2, \ldots, x_n; r)$ and $(x_1, x_2, \ldots, x_n; s)$ are two one relator presentations of G, then there is an α in A such that $r\alpha = s$ or $r\alpha = s^{-1}$.

We show that this conjecture is equivalent to the statement; if G is a one relator group then every generating n-tuple of G associated with a one relator presentation lies in the same T-system.

Thus let g and h be generating n-tuples of G associated with the presentations $(x_1, x_2, \ldots, x_n; r)$ and $(x_1, x_2, \ldots, x_n; s)$ respectively. Put $R(\mathrm{g}) = \langle r \rangle^F$ and $R(\mathrm{h}) = \langle s \rangle^F$.

If g and h lie in the same T-system then there is an α in A and a β in B such that $\mathrm{g}\alpha\beta = \mathrm{h}$. But

$$R(\mathrm{h}) = R(\mathrm{g}\alpha\beta) = R(\mathrm{g}\alpha) = R(\mathrm{g})\alpha^{-1}$$

(see our comments in §2.3) so that by Theorem 4.11 on p. 261 of [12] we have s^{ε} and $r\alpha^{-1}$ conjugate in F, where $\varepsilon = \pm 1$. Thus there is an α' in A such that $r\alpha' = s$ or $r\alpha' = s^{-1}$.

Conversely, if $r\alpha = s^{\varepsilon}$ for some α in A and $\varepsilon = \pm 1$ then $R(\mathfrak{h}) = \langle s \rangle^F = \langle r\alpha \rangle^F = \langle r \rangle^F \alpha = R(\mathfrak{g})\alpha = R(\mathfrak{g}\alpha^{-1})$. Thus there is a β in B such that $\mathfrak{h} = \mathfrak{g}\alpha^{-1}\beta$ so that $\mathfrak{g}$ and $\mathfrak{h}$ lie in the same T-system.

Now Magnus's conjecture has been disproved by J. McCool and A. Pietrowski in [13] who find counterexamples amongst the groups $G_{k,l}$ which have the presentation $(x, y; x^{-k}y^{l})$. Consequently the groups $G_{k,l}$ provide examples of groups with more than one T-system of generating pairs associated with one relator presentations. For example, the generating pair (a, b) of $G_{5,2}$ associated with the presentation $(x, y; x^{-5}y^{2})$ is in a different T-system from the generating pair (a^2, b) associated with the presentation $(x, y; x^{-1}(y^2x^{-2})^2)$.

We study in §2.6 the T-systems of the group $\mathrm{gp}(a, b; b^{-1}a^2b = a^3)$. It will follow from Theorem 2.6.6 that the generating pair (a^2, b) associated with the presentation $(x, y; x^{-1}[x, y]^2)$ and the generating pair (a, b) lie in different T-systems. Thus the group $\mathrm{gp}(a, b; b^{-1}a^2b = a^3)$ provides another counterexample to Magnus's conjecture.

The following (modified) conjecture would seem to be hopeful.

Conjecture. Let G be an n-generator one relator group and let $\mathfrak{g}$ and $\mathfrak{h}$ be generating n-tuples associated with one relator presentations of G. Then either g is in the same T-system as a generating n-tuple $\mathsf{h}' = \left(h_1^{s_1}, h_2^{s_2}, \ldots, h_n^{s_n}\right)$ where $s_1, s_2, \ldots, s_n$ are positive integers, or, h is in the same T-system as a generating n-tuple $\mathsf{g}' = \left(g_1^{t_1}, g_2^{t_2}, \ldots, g_n^{t_n}\right)$ where $t_1, t_2, \ldots, t_n$ are positive integers.

2.5 Some metabelian groups with one T-system

The main theorem proved in this section is Theorem 2.5.1 which states that a two generator group G having an abelian normal subgroup N such that G/N is infinite cyclic has one T-system of generating pairs.

In Theorem 2.5.2 we apply this theorem to the groups $\mathrm{gp}\left(a, b;\, b^{-1}ab = a^r\right)$ where r is an integer. We also show that splitting metacyclic groups have one T-system in Theorem 2.5.3.

A related result to Theorem 2.5.1 is a result of M.J. Dunwoody (see Theorem 4.10 of [7]). This states that a two generator metabelian group whose commutator quotient group is free abelian of rank 2 has one T-system. However, not all two generator metabelian groups have one T-system as M.J. Dunwoody has found (in [6]) examples of finite p-groups, nilpotent of class 2, having arbitrary numbers of T-systems.

Finally in Theorem 2.5.4 we show that $\mathrm{gp}\left(a, b;\, b^{-1}ab = a^4\right)$ is an example of a one relator group with more than one A-class but one T-system.

2.5.1 THEOREM. *Let G be a two generator group with an abelian normal subgroup N such that G/N is an infinite cyclic* *group. Then G is a metabelian group with one T-system of generating pairs.*

Proof. According to Lemma 2.2.5 there is a generating pair (a, b) of G with a in N and G/N generated by bN. Also, any other generating pair lies in the same T-system as a pair (c, bd) where c and d belong to N.

Since a belongs to N we have $\langle a\rangle^G$ contained in N. Also, as G/N is generated by bN, it follows that any element of N has zero exponent sum on b and consequently belongs to $\langle a\rangle^G$. Thus $N = \langle a\rangle^G$ and N is generated by the elements a^{b^i} for every integer i.

Now G/N is infinite cyclic and generated by bN so it follows that the relations of G can be written in the form $a^t = w(a, b)$, where t is some integer and $w(a, b)$ is a word with zero exponent sum on b. Here $w(a, b) = \prod_j a^{l_j b^j}$ for some integers l_j. Further c, which is an element of N, can be written in the form $c = \prod_i a^{k_i b^i}$ for some integers k_i.

We note that if n is any element of N and s an integer then $n^{(bd)^s} = n^{b^s d^{b^{s-1}} d^{b^{s-2}} \cdots d} = n^{b^s}$, since n^{b^s} and d^{b^r}, for any integer r, are elements of N which is abelian.

Consider $w(c, bd)$ which is the word obtained from $w(a, b)$ by replacing every occurrence of a by c and every occurrence of b by bd.

Then

$$w(c,\ bd) = \prod_j c^{l_j(bd)^j}$$

$$= \prod_j c^{l_j b^j} \quad \text{by our comment above,}$$

$$= \prod_j \left(\prod_i a^{k_i b^i} \right)^{l_j b^j}$$

$$= \prod_i \left(\prod_j a^{l_j b^j} \right)^{k_i b^i} \quad \text{since } N \text{ is abelian,}$$

$$= \prod_i \big(w(a,\ b)\big)^{k_i b^i} .$$

It follows that

$$c^{-t} w(c,\ bd) = \left(\prod_i a^{-k_i b^i} \right)^t w(c,\ bd)$$

$$= \prod_i \big(a^{-t} w(a,\ b)\big)^{k_i b^i} = 1 ,$$

using again the fact that N is abelian.

Consequently, if we define a mapping ϑ of G into G by $a\vartheta = c$ and $b\vartheta = bd$ we see that ϑ extends to an ependomorphism of G. Now since G/N is abelian N contains the derived group G'. But N is abelian so that G' is. Thus G is metabelian. A theorem of P. Hall (Theorem 1 of [10]) implies that G is residually finite and hence, by 41.44 of [17], a Hopf group. It follows that ϑ is an automorphism of G. We conclude that G has one T-system. //

As a corollary to this theorem we have,

2.5.2 THEOREM. *The group* $G = \mathrm{gp}(a,\ b;\ b^{-1}ab = a^r)$ *is a metabelian group with one* T*-system of generating pairs.*

Proof. For any integer s we have $b^{-s}ab^{s} = a^{r^s}$. Also

$$ab^{s}ab^{-s} = b^{s}a^{r^s}ab^{-s}$$
$$= b^{s}aa^{r^s}b^{-s}$$
$$= b^{s}ab^{-s}b^{s}a^{r^s}b^{-s}$$
$$= b^{s}ab^{-s}a \ .$$

Thus $\langle a\rangle^G$ is an abelian group. Also $G/\langle a\rangle^G$ is an infinite cyclic group so that Theorem 2.5.1 gives the required result. //

In our next theorem we show that splitting metacyclic groups have one T-system.

2.5.3 THEOREM. *The metacyclic group*

$$G = \mathrm{gp}\left(a,\ b;\ a^m = b^n = 1,\ b^{-1}ab = a^r\right)$$

where r, m *and* n *are integers with* $r^n \equiv 1$ *modulo* m *and* $(m, r\text{-}1) = 1$ *has one* T*-system of generating pairs.*

Proof. Let $N = \langle a\rangle^G$. Since $ba^{r}b^{-1} = a$ we have $bab^{-1} = ba^{r^n}b^{-1} = a^{r^{n-1}}$. It follows that the cyclic subgroup of G generated by a is normal and thus equal to N.

By Lemma 2.2.5 any generating pair of G lies in the same T-system as a pair (c, bd) where c and d belong to N. Thus $(c, bd) = \left(a^k,\ ba^l\right)$ for some integers k and l.

We prove first that for any integer l we have $\left(ba^l\right)^n = 1$. But

$$\begin{aligned}(ba^l)^n &= b^n(a^l)^{b^{n-1}}(a^l)^{b^{n-2}} \ldots (a^l)\\ &= b^n\left(a^{b^{n-1}}a^{b^{n-2}} \ldots a\right)^l\\ &= b^n\left(a^{r^{n-1}+r^{n-2}+\ldots+1}\right)^l .\end{aligned}$$

Thus it remains to show that $r^{n-1} + r^{n-2} + \ldots + 1 \equiv 0$ modulo m. But from $r^n \equiv 1$ modulo m we have $(r-1)\left(r^{n-1} + \ldots + 1\right) \equiv 0$ modulo m and as $(r-1, m) = 1$ we have $r^{n-1} + \ldots + 1 \equiv 0$ modulo m. Now $\left(a^k\right)^m = 1$ and $a^{-l}b^{-1}a^kba^l = \left(a^k\right)^r$. Thus if θ is a mapping of G defined by $a\theta = a^k$ and $b\theta = ba^l$ then θ extends to an ependomorphism of G. Since G is finite we conclude that θ is an automorphism thus completing the proof. //

Remark. It follows from Theorem 2.5.2 that every generating pair of $\mathrm{gp}\left(a, b; b^{-1}ab = a^r\right)$ is associated with a one relator presentation.

According to B.H. Neumann [16] if $(m, n) = 1$ then the group $\mathrm{gp}\left(a, b; a^m = b^n = 1, b^{-1}ab = a^r\right)$ has a presentation on the same generating pair $\mathrm{gp}\left(a, b; a^m = b^n, b^{-1}a^sb = a^{s-1}\right)$ where s is a solution of $(r-1)s \equiv -1$ modulo m. It follows from Theorem 2.5.3 that every generating pair of this group is associated with a two relator presentation.

We conclude with some remarks on A-classes of certain groups.

2.5.4 LEMMA. *Let G be a two generator group and ψ the natural epimorphism of G onto G/G'. If (a, b) and (c, d) are generating pairs of G such that $(a\psi, b\psi)$ and $(c\psi, d\psi)$ lie in different A-classes of G/G' then (a, b) and (c, d) lie in different A-classes of G.*

Proof. Suppose that there is an α in A such that $(a, b)\alpha = (c, d)$. But then $(a, b)\alpha\psi = (c, d)\psi$, and using Lemma 2.1.1 we have $(a\psi, b\psi)\alpha = (a, b)\psi\alpha = (a, b)\alpha\psi = (c, d)\psi = (c\psi, d\psi)$ which is a contradiction. //

It is not hard to show, for example, that the generating pairs (e, f) and (e^2, f) of the group $\mathrm{gp}(e, f;\ e^5 = [e, f] = 1)$ lie in different A-classes. If α belongs to A and $(e, f)\alpha = (e^2, f)$ where $(e, f)\alpha = (e^{s_1}f^{t_1}, e^{s_2}f^{t_2})$ then $s_1 \equiv 2$ modulo 5 , $t_1 = 0$ and $t_2 = 1$. But since $s_1t_2 - t_1s_2 = \pm 1$ we have $s_1 = \pm 1$ which is a contradiction.

Combining Lemma 2.5.4 with this remark we have that $\mathrm{gp}(a, b;\ a^5 = 1)$ has generating pairs (a, b) and (a^2, b) which lie in different A-classes. A similar statement is also true of the group $\mathrm{gp}(c, d;\ c^5 = d^5 = (cd)^5 = 1)$.

In a recent paper [18], N. Purzitsky and G. Rosenberger claim to have shown that every two generator Fuchsian group has one A-class of generating pairs. Since the group $\mathrm{gp}(a, b;\ a^5 = 1)$ has at least two they appear to be mistaken*. However since their work contains much detailed information on generators of Fuchsian groups, hopefully they have shown that any other generating pair of a two generator Fuchsian group is in the same A-class as a generating pair (a^s, b^t) for some integers s and t.

In [19] E.S. Rapaport considered the group of Listing's knot and showed that it has more than one A-class of generating pairs. The group of Listing's knot is a two generator one relator group. It

* This was noticed by S.J. Pride.

was also shown there that the group has one T-system of generating pairs associated with one relator presentations. I do not know whether this group has one T-system of generating pairs or not.

In Theorem 2.5.4 we show that the group $\text{gp}(a, b; b^{-1}ab = a^4)$ does not have one A-class, although, by Theorem 2.5.2 it has one T-system of generating pairs.

2.5.4 THEOREM. *The group* $G = \text{gp}(a, b; b^{-1}ab = a^4)$ *has more than one* A*-class of generating pairs.*

Proof. We show that G has an outer automorphism θ such that $(a, b)\theta = (a^2, b)$ and that there is no α in A such that $(a, b)\alpha = (a^2, b)$.

Firstly a^2 and b generate G since $a = (ba^2b^{-1})^2$. Also, if θ is a mapping of G defined by $a\theta = a^2$ and $b\theta = b$ then as $b^{-1}a^2b = (a^4)^2 = (a^2)^4$ we see that θ extends to an ependomorphism of G. However, since $b^{-1}(ba^2b^{-1})b = a^2 = (ba^2b^{-1})^4$, we see that the mapping θ' of G defined by $a\theta' = ba^2b^{-1}$ and $b\theta' = b$ also extends to an ependomorphism of G which is clearly inverse to θ. Thus θ is an automorphism.

If θ is inner then for some w in G we have $a^w = a^2$, or, $a = [a, w]$. But this implies that G/G' is cyclic, whereas in fact G/G' is the direct product of a cyclic group of order 3 and an infinite cyclic group.

Suppose that there is an α in A such that $(a, b)\alpha = (a^2, b)$. By Theorem 2.2.6 we have that $[a, b]^\varepsilon$, where $\varepsilon = \pm 1$, and $[a^2, b]$ are conjugate in G. That is there is an element v in G

such that $(a^3)^{\varepsilon} = (a^6)^{v}$.

Now $v = b^n c$ where n is some integer and c belongs to $\langle a \rangle^G$. But $\langle a \rangle^G$ is abelian (see the proof of Theorem 2.3.2) so that $(a^3)^{\varepsilon} = (a^6)^{b^n}$.

If n is a negative integer then we have

$\left(a^{4^{-n}3}\right)^{\varepsilon} = (a^3)^{\varepsilon b^{-n}} = a^6$, so that a has finite order when $\varepsilon = +1$ or $\varepsilon = -1$. If n is a non-negative integer then

$(a^3)^{\varepsilon} = (a^6)^{b^n} = a^{6 \cdot 4^n}$ so that a has finite order when $\varepsilon = +1$ or $\varepsilon = -1$.

However from Theorem 4.12 of [12] we see that G cannot have any element of finite order. This is a contradiction.

It follows that (a, b) and (a^2, b) lie in different A-classes. //

2.6 A one relator group with an infinity of T-systems

In this section we show that the group $G = \mathrm{gp}\left(a, b;\ b^{-1}a^2b = a^3\right)$ has an infinite number of T-systems of generating pairs.

The presentation of G associated with the generating pair $\left(a^{2^n}, b\right)$, for each positive integer n , is shown to be $\left(x, y;\ x^{-1}[x, y]^2, \left[x, x^{y^n}\right]\right)$. We also show that for different non-negative integers n the generating pairs $\left(a^{2^n}, b\right)$ lie in different T-systems.

We first need some preliminary results. Let F denote the free group generated by x and y.

2.6.1 LEMMA. *The word* $x^{-1}[x, y]^2$ *is equivalent in* F *to the words* $y^{-1}x^2yx^{-3}$ *and* $[x, [x, y]]$.

Proof. It is enough to show that if c and d are arbitrary elements of a group H then $c = [c, d]^2$ if and only if $d^{-1}c^2d = c^3$ and $[c, [c, d]] = 1$.

But if $c = [c, d]^2$ then $[c, [c, d]] = [[c, d]^2, [c, d]] = 1$. Thus

$$1 = c^{-1}[c, d]^2 = c^{-1}[c, d]c^{-1}d^{-1}cd = c^{-2}[c, d]d^{-1}cd$$
$$= c^{-3}d^{-1}cdd^{-1}cd = c^{-3}d^{-1}c^2d .$$

However, clearly the argument is reversible and this completes the proof. //

2.6.2 LEMMA. *Let* H *be the group*

$$\mathrm{gp}\left\{c, d; c = [c, d]^2, \left[c, c^{d^n}\right] = 1\right\}$$

where n *is some positive integer. Then the relations* $\left[c, c^{d^j}\right] = 1$ *for* $j = 1, 2, \ldots, n$ *hold in* H.

Proof. We use the notation that $c_i = c^{d^i}$ for each integer i. From Lemma 2.6.1, as $c = [c, d]^2$, we have $c_1^2 = c_0^3$ and $[c_0, c_1] = 1$. Now $c_1^2 = c_0^3$ implies that $c_r^{2^r} = \left(c_0^{2^r}\right)^{d^r} = c_0^{3^r}$, for any positive integer r, so that $\left[c_0, c_r^{2^r}\right] = 1$. Also conjugation of $c_r^{2^r} = c_0^{3^r}$ by d^{n-r} implies that $c_n^{2^r} = c_{n-r}^{3^r}$, so that

$$1 = [c_0, c_n] = \left[c_0, c_n^{2^r}\right] = \left[c_0, c_{n-r}^{3^r}\right] .$$

Suppose now that j is an integer with $j = 1, 2, \ldots, n$. Then

$$1 = \left[c_0, c_j^{2^j}\right] = \left[c_0, c_j^{3^{n-j}}\right]$$ from the above. But $\left(2^j, 3^{n-j}\right) = 1$,

so that $[c_0, c_j] = 1$.

2.6.3 LEMMA. *Let* n *be a positive integer. The set of words*

$$\left\{x^{-1}[x, y]^2, \left[[x, y], [x, y]^{y^j}\right]; j = 1, 2, \ldots, n\right\}$$

is equivalent in F *to the set of words* $\left\{x^{-1}[x, y]^2, \left[x, x^{y^{n+1}}\right]\right\}$.

Proof. Let c and d be elements of an arbitrary group H . We use c_i to denote the element c^{d^i} for each integer i . We note that $[c, d]^{d^j} = \left(c^{-1}c^d\right)^{d^j} = c^{-d^j}c^{d^{j+1}} = c_j^{-1}c_{j+1}$ for any integer j . It is enough to show that $c = [c, d]^2$ and $\left[c_0^{-1}c_1, c_j^{-1}c_{j+1}\right] = 1$ for $j = 1, 2, \ldots, n$ if and only if $c = [c, d]^2$ and $[c, c_{n+1}] = 1$.

We note the following identity,

$$[uv, wz] = [u, z]^v[v, z][u, w]^{vz}[v, w]^z$$

(see 33.34 (1) of [17]). Thus

$$\left[c_0^{-1}c_1, c_j^{-1}c_{j+1}\right] = \left[c_0^{-1}, c_{j+1}\right]^{c_1}\left[c_1, c_{j+1}\right]\left[c_0^{-1}, c_j^{-1}\right]^{c_1c_{j+1}}\left[c_1, c_j^{-1}\right]^{c_{j+1}}$$

$$= \left[c_0^{-1}, c_{j+1}\right]^{c_1}\left[c_0, c_j\right]^d\left[c_0^{-1}, c_j^{-1}\right]^{c_1c_{j+1}}\left[c_0, c_{j-1}\right]^{dc_{j+1}} .$$

This implies that for any integer j ,

(i) $\left[c_0^{-1}c_1, c_j^{-1}c_{j+1}\right]$ belongs to

$$\langle\{[c_0, c_i];\ i = j-1, j, j+1\}\rangle^G \text{ , and}$$

(ii) $\left[c_0, c_{j+1}\right]$ belongs to

$$\left\langle\left\{\left[c_0^{-1}c_1, c_j^{-1}c_{j+1}\right], \left[c_0, c_{j-1}\right], \left[c_0, c_j\right]\right\}\right\rangle^G .$$

Suppose that $c = [c, d]^2$ and $\left[c, c_{n+1}\right] = 1$. By Lemma 2.6.2 we have $\left[c_0, c_j\right] = 1$ for $j = 1, 2, \ldots, n+1$. From (i) we conclude that $\left[c_0^{-1}c_1, c_j^{-1}c_{j+1}\right] = 1$ for $j = 1, 2, \ldots, n$.

Suppose now that $\left[c_0^{-1}c_1, c_j^{-1}c_{j+1}\right] = 1$ for $j = 1, 2, \ldots, n$ and $c = [c, d]^2$.

We prove that $\left[c_0, c_{n+1}\right] = 1$ by induction on n.

When $n = 1$, since $c = [c, d]^2$, we have $\left[c_0, c_1\right] = 1$. From (ii) we conclude that $\left[c_0, c_2\right] = 1$.

Suppose now that $\left[c_0, c_j\right] = 1$ for $j = 1, 2, \ldots, n+1$ and that $\left[c_0^{-1}c_1, c_n^{-1}c_{n+2}\right] = 1$. From (ii) we conclude that $\left[c_0, c_{n+2}\right] = 1$ thus completing the induction step. //

2.6.4 THEOREM. *Let G be the group* $\mathrm{gp}\left(a, b;\ b^{-1}a^2b = a^3\right)$. *For each positive integer n the pair $\left(a^{2^n}, b\right)$ is a generating pair of G associated with a presentation* $\left\{x, y;\ x^{-1}[x, y]^2, \left[x, x^{y^n}\right]\right\}$ *of G.*

Proof. For any positive integer r we have

$$\left[a^{2^r}, b\right] = a^{-2^r} b^{-1} a^{2^r} b = a^{-2^r} a^{3-2^{r-1}} = a^{2^{r-1}}.$$

Thus if n is a positive integer then $a = \left[a^{2^n}, nb\right]$ so that $\left(a^{2^n}, b\right)$ is a generating pair of G.

We prove the theorem by induction on n.

Now (a, b) is associated with a presentation $\left(x, y;\ y^{-1}x^2yx^{-3}\right)$. By applying Tietze transformations (see §1.5 of [12]) we see that

$$\left(x, y, z;\ z^{-1}x^2, x^{-1}\left[x^2, y\right]\right)$$

and also $\left(z, y;\ z^{-1}[z, y]^2\right)$ are presentation of G; the last being associated with the pair $\left(a^2, b\right)$, which proves the assertion for $n = 1$.

Assume now that the assertion is true for some positive integer n. Thus we have a generating pair $\left(a^{2^n}, b\right)$ associated with the presentation $\left(x, y;\ x^{-1}[x, y]^2, \left[x, x^{y^n}\right]\right)$ of G.

Now by applying appropriate Tietze transformations we see that the following are presentations of G;

$\left(x, y;\ x^{-1}[x, y]^2, \left[x, x^{y^j}\right] \text{ for } j = 1, 2, \ldots, n\right)$ by Lemma 2.6.2,

$\left(x, y;\ x^{-1}\left[x^2, y\right], \left[x, x^{y^j}\right] \text{ for } j = 1, 2, \ldots, n\right)$ by Lemma 2.6.1,

$$\left(x, y, z;\ z^{-1}x^2, x^{-1}\left[x^2, y\right], \left[x, x^{y^j}\right] \text{ for } j = 1, 2, \ldots, n\right),$$

$$\left(z, y;\ z^{-1}[z, y]^2, \left[[z, y], [z, y]^{y^j}\right] \text{ for } j = 1, 2, \ldots, n\right),$$

and

$$\left\{z, y;\ z^{-1}[z, y]^2,\ \left[z, z^{y^{n+1}}\right]\right\} \text{ by Lemma 2.6.3.}$$

The last presentation is associated with the pair $\left(a^{2^{n+1}}, b\right)$ and this completes the induction step and the proof of the theorem. //

The proof of the following lemma is modelled on the style of proof used on p. 261 of [12] to establish the non-Hopf property of the group $\mathrm{gp}\left(a, b;\ b^{-1}a^2b = a^3\right)$.

2.6.5 LEMMA. *Let m and n be non-negative integers with $m > n$, and let G be the group $\mathrm{gp}\left(a, b;\ b^{-1}a^2b = a^3\right)$. Then there is no automorphism θ of G such that $\left(a^{2^m}, b\right)\theta = \left(a^{2^n}, b\right)$.*

Proof. Firstly $\left[a^{2^r}, b\right] = a^{2^{r-1}}$ for any positive integer r. Thus as $m > n$ we have $\left[a^{2^m}, (n+1)b\right] = a^{2^{m-n-1}}$ and consequently

$$\left[a^{2^m}, (n+1)b\right]^{2^{n+1}} = a^{2^m}.$$

Suppose that there is an automorphism θ with $\left(a^{2^m}, b\right)\theta = \left(a^{2^n}, b\right)$. Then, as $\left[a^{2^m}, (n+1)b\right]^{2^{n+1}} = a^{2^m}$, we have $\left[a^{2^n}, (n+1)b\right]^{2^{n+1}} = a^{2^n}$. But $a = \left[a^{2^n}, nb\right]$ so that $[a, b]^{2^{n+1}} = a^{2^n}$. We show that this is a contradiction.

Let $N = \langle a\rangle^G$. Then N has $\left\{a^{b^i};\ i \in \mathbb{Z}\right\}$ as a generating set. We denote by a_i the element a^{b^i} for each integer i. According to [12] on p. 261,

$$N = \mathrm{gp}\left\{\ldots, a_{-1}, a_0, a_1, \ldots ; \ldots, a_{-1}^3 = a_0^2, a_0^3 = a_1^2, \ldots\right\} .$$

Now there is an integer t such that $2^n = 3t \pm 1$ so that we may write $a^{2^n}[a, b]^{-2^{n+1}}$ as $a_0^{3t}a_0^{\varepsilon}\left(a_1^{-1}a_0\right)^{2^{n+1}}$ where $\varepsilon = \pm 1$. This is an element of $N_0 = \mathrm{sbgp}\left(a_0, a_1\right) = \mathrm{gp}\left(a_0, a_1 ; a_0^3 = a_1^2\right)$ (see [12], p. 261). Choosing a_0 and a_0^{-1} as the representatives of $\mathrm{sbgp}\left(a_0\right)$ modulo $\mathrm{sbgp}\left(a_0^3\right)$ and a_1^{-1} as the representative of $\mathrm{sbgp}\left(a_1\right)$ modulo $\mathrm{sbgp}\left(a_0^3\right)$, by the solution to the word problem for free products with amalgamation (see Theorem 4.4 of [12]), we see that $a_0^{3t}a_0^{\varepsilon}\left(a_1^{-1}a_0\right)^{2^{n+1}}$ does not define the identity in N_0 and hence in G for any non-negative integer n. This is the desired contradiction. //

2.6.6 THEOREM. *Let* G *be the group* $\mathrm{gp}\left(a, b ; b^{-1}a^2b = a^3\right)$. *For different non-negative integers* n *the generating pairs* $\left(a^{2^n}, b\right)$ *lie in different* T*-systems of* G.

Proof. By Theorem 2.6.4 the pairs $\left(a^{2^n}, b\right)$ are generating pairs of G. Let m and n be non-negative integers with $m > n$, and we make the assumption that $\left(a^{2^n}, b\right)$ and $\left(a^{2^m}, b\right)$ lie in the same T-system.

Let $N = \langle a\rangle^G$. Then, as $a = \left[a^2, b\right]$, we have $N = G'$. In particular N is characteristic in G.

Let A denote the automorphism group of the free group of rank 2 and let B denote the automorphism group of G. From Lemma 2.1.1

we see that the elements of $\overline{A}$ and $\overline{B}$ centralize each other. Consequently there is an α in A and a β in B such that

$(a^{2^n}, b)\alpha = (a^{2^m}, b)\beta$.

Now bN generates the infinite cyclic group G/N and as N is characteristic we see that $b\beta = b^{\varepsilon}d$ where $\varepsilon = \pm 1$ and d belongs to N . Further, since N is characteristic, $a^{2^m}\beta = c$ an element of N . Thus $(a^{2^m}, b)\beta = (c, b^{\varepsilon}d)$.

Consequently $(a^{2^n}, b)\alpha = (c, b^{\varepsilon}d)$ where $\varepsilon = \pm 1$ and c, d belong to N . Let ψ denote the natural epimorphism of G onto G/N . By Lemma 2.6.1 we have

$$\begin{aligned}((a^{2^n}, b)\alpha)\psi &= ((a^{2^n}, b)\psi)\alpha \\ &= (b^0N, bN)\alpha ,\end{aligned}$$

whereas $(c, b^{\varepsilon}d)\psi = (b^0N, b^{\varepsilon}N)$. Thus $(b^0N, bN)\alpha = (b^0N, b^{\varepsilon}N)$. By Lemma 2.2.2 we have $\alpha = \overline{\alpha}\gamma$. Here $\overline{\alpha}$ is an element of A given by $x_1\overline{\alpha} = x_1^{\varepsilon'}$ and $x_2\overline{\alpha} = x_1^s x_2^{\varepsilon}$ where $\varepsilon' = \pm 1$, s is an integer, and γ is an inner automorphism of the free group of rank 2 .

It follows that $(a^{2^n}, b)\overline{\alpha}\gamma = (a^{2^m}, b)\beta$, and in view of Lemma

we have $(a^{2^n}, b)\overline{\alpha} = (a^{2^m}, b)\beta'$, where β' is an element of B .

Now it is easy to see that there are automorphisms η_1 and η_2 of G such that $(a, b)\eta_1 = (a^{-1}, b)$ and $(a, b)\eta_2 = (a, a^r b)$ for any integer r .

Thus $(a^{2^n}, b)\alpha = (a^{2^n\varepsilon'}, a^{2^n s}b^{\varepsilon}) = (a^{2^n}, b^{\varepsilon})\eta$ for some η in B.

Consequently we have that $(a^{2^n}, b^{\varepsilon})\theta = (a^{2^m}, b)$ for some θ in B.

When $\varepsilon = 1$ it was shown in Lemma 2.6.5 that no such automorphism θ exists. Suppose that $\varepsilon = -1$. Then, since $\left(a^{2^m}\right)^{4b} = a^{2^m 6}$ and θ is an automorphism, we have $\left(a^{2^n}\right)^{4b^{-1}} = a^{2^n 6}$. Thus $a^{2^{n+2}} = \left(a^{2^n 6}\right)^b = a^{9 \cdot 2^n}$ so that $a^{2^n 5} = 1$. However G has no element of finite order (see, for example, Theorem 4.12 of [12]). Thus we have the desired contradiction. //

For each positive integer i let

$$G_i = \text{gp}\left(a_i, b_i; a_i = [a_i, b_i]^2, \left[a_i, a_i^{b_i^i}\right] = 1\right)$$

and let $G_0 = \text{gp}\left(a_0, b_0; b_0^{-1}a_0^2 b_0 = a_0^3\right)$. Then each G_i is isomorphic to G_0 and for each non-negative integer n there is an epimorphism φ_n of G_n onto G_{n+1}.

By Lemma 2.6.5 each φ_n has a non-trivial kernel. We may form the direct limit of the G_n which can be presented as the group

$H = \text{gp}\left(a, b; a = [a, b]^2, \left[a, a^{b^i}\right] = 1 \text{ for } i = 1, 2, \ldots\right)$. The group H is metabelian and has an infinite cyclic commutator quotient group so that H, by Theorem 2.5.1, has only one T-system. However each of the groups G_n have, by Theorem 2.6.6, an infinity of T-systems.

In [13] J. McCool and A. Pietrowski asked the still unsolved problem of whether there can exist an n-generator one relator group with an infinite number of T-systems of generating n-tuples associated with one relator presentations.

Our study of the group $\mathrm{gp}(a, b;\ b^{-1}a^2b = a^3)$ provides the solution to the less ambitious question of whether there exist one relator groups with an infinity of T-systems. Some time after we had done this work Dr M.F. Newman received a preprint of a (still unpublished) paper by M.J. Dunwoody and A. Pietrowski who also find an example of a one relator group with an infinite number of T-systems. They show that the trefoil knot group $\mathrm{gp}(a, b;\ a^2 = b^3)$ has generating pairs (a^{2i+1}, b^{3i+1}), for each non-negative integer i, which for different i lie in different T-systems. When i is non-zero the presentations associated with the generating pairs (a^{2i+1}, b^{3i+1}) require at least two relators.

CHAPTER III

ISOMORPHISMS OF CYCLIC EXTENSIONS OF FREE GROUPS

3.1 The first isomorphism criteria

Let $X(r)$, for r an integer greater than 1 , denote the class of all groups which are two generator one relator and an extension of a free group of rank r by a free group of rank one.

Suppose that a group G can be presented in the form

$$G = \mathrm{gp}\left(a,\, b;\, a^{b^r} = ua^{\varepsilon}u'\right) \qquad (3.1)$$

where $\varepsilon = \pm 1$ and u, u' belong to the (free) group generated by $a^b, a^{b^2}, \ldots, a^{b^{r-1}}$ for some integer r greater than 1 . A presentation of the form (3.1) is called a *standard presentation* of G .

The following theorem shows that every group in $X(r)$ has at least one standard presentation. In Lemma 3.1.2 we will show that every group with a standard presentation (3.1), for some r , belongs to $X(r)$.

3.1.1 THEOREM. *Any generating pair associated with a one relator presentation of a group in* $X(r)$ *is in the same A-class as a generating pair associated with a standard presentation.*

For a proof of this theorem we refer to the proof of Lemma 1 of E.S. Rapaport [19]. The situation considered there is when G is an extension by a free group of rank one of its derived group which is free of rank two. It is shown that any generating pair associated with a one relator presentation is in the same A-class as a generating pair (a, b) where a belongs to G' and bG' generates G/G' . By a Schreier rewriting process generators of G'

are found. Then, using the fact that G' is free of rank two, the result is obtained. Now this proof does not rely on the fact that G' is the commutator subgroup but only upon the facts that G' is free of finite rank and G/G' is free of rank one. If in the proof we replace G' by a free normal subgroup H of rank r such that G/H is free of rank one then a proof of Theorem 3.1.1 can be obtained.

3.1.2 LEMMA. *Let G be a group with a standard presentation (3.1). Then $\langle a\rangle^G$ is a free group of rank r freely generated by $a, a^b, \ldots, a^{b^{r-1}}$.*

Proof. The word $a^{-b^r}ua^{\varepsilon}u'$ has zero exponent sum on b. We apply the considerations involved in Case 2 of Theorem 4.10 of [12]. Thus let a_i denote the element a^{b^i} and r_i denote the word $\left(a^{-b^r}ua^{\varepsilon}u'\right)^{b^i}$ for each integer i. Then

$\langle a\rangle^G = \mathrm{gp}\left(\ldots, a_{-1}, a_0, a_1, \ldots; \ldots, r_{-1} = 1, r_0 = 1, r_1 = 1, \ldots\right)$.

Now $\mathrm{gp}\left(a_0, a_1, \ldots, a_r; r_0 = 1\right)$ is a subgroup of $\langle a\rangle^G$ as was demonstrated in the proof of Case 2 of Theorem 4.10 of [12]. But $r_0 = 1$ can be written, in G, as $a^{b^r} = ua^{\varepsilon}u'$, or, in other words a_r can be expressed in terms of $a_0, a_1, \ldots, a_{r-1}$. Thus $\mathrm{gp}\left(a_0, a_1, \ldots, a_r; r_0 = 1\right)$ is the free group generated by $a_0, a_1, \ldots, a_{r-1}$. Since the elements $a, a^b, \ldots, a^{b^{r-1}}$ generate $\langle a\rangle^G$ we conclude that $\langle a\rangle^G$ itself is a free group freely generated by these elements. //

We remark that although Theorem 3.1.1 states that every generating pair associated with a one relator presentation is in the same A-class as a generating pair associated with a standard presentation, not every generating pair associated with a standard presentation is in the same A-class.

It is the purpose of this chapter to examine the relationship between the various standard presentations of a group in $X(r)$.

We now give some general remarks about a group with standard presentation (3.1).

Firstly if $G = \mathrm{gp}\left(a, b; a^{b^r} = ua^{\varepsilon}u'\right)$ then we may write

$$G = \mathrm{gp}\Big(a, d; a^{d^r} = \left[\left(u\left(a^{d^{(r-1)}}, \ldots, a^{d}\right)\right)^{-1} a\left(u'\left(a^{d^{(r-1)}}, \ldots, a^{d}\right)\right)^{-1}\right]^{\varepsilon}\Big) \qquad (3.2)$$

where $d = b^{-1}$ and $u = u\left(a, a^b, \ldots, a^{b^{r-1}}\right)$,

$u' = u'\left(a, a^b, \ldots, a^{b^{r-1}}\right)$. Thus (3.2) is a standard presentation for G with generating pair $\left(a, b^{-1}\right)$.

Let s' be the exponent sum of $a^{-b^r}ua^{\varepsilon}u'$ on a. When $s' = 0$ the commutator quotient group G/G' is free abelian of rank two and by Theorem 4.10 of [7] we have that G/G'' has one T-system. When s' is a non-zero integer it is clear that G/G' is the direct product of a cyclic group of order $|s'|$ generated by aG', and an infinite cyclic group generated by bG'. We call $s = |s'|$ the *torsion number* of G. The torsion number s is independent of the standard presentation of G.

3.1.3 LEMMA. *If G has a non-zero torsion number s then $\langle a\rangle^G$ is the unique normal subgroup H of G such that G/H is infinite cyclic.*

Proof. Suppose that M is a normal subgroup of G such that G/M is an infinite cyclic group. Thus G' is contained in M, and since a^s belongs to G' we must have that a belongs to M; for otherwise aM would be an element of finite order in G/M. It follows that $\langle a\rangle^G$ is contained in M. But if this containment is proper then as $G/M \cong \left(G/\langle a\rangle^G\right)/\left(M/\langle a\rangle^G\right)$, and both G/M and $G/\langle a\rangle^G$ are infinite cyclic groups, we have a contradiction. //

Remark. We use the following notation. If G is a group with a standard presentation (3.1) then we define $\sigma_i(a, b)$ to be the exponent sum of $a^{-b^r}ua^{\varepsilon}u'$ on $a^{b^{i-1}}$, where $a^{-b^r}ua^{\varepsilon}u'$ is considered here as a word in the free group on $a, a^b, \ldots, a^{b^r}$ for $i = 1, 2, \ldots, r$.

By (3.2), since $\left(a, b^{-1}\right)$ is associated with a standard presentation, $\sigma_i\left(a, b^{-1}\right)$ for $i = 1, 2, \ldots, r$ also has meaning under this definition.

3.1.4 LEMMA. *Let $H = \langle a\rangle^G$ and suppose that the torsion number s of G is non-zero. The characteristic polynomial of the action of b on H/H' is*

$$(-1)^r\left[\vartheta^r - \sigma_r(a, b)\vartheta^{r-1} - \sigma_{r-1}(a, b)\vartheta^{r-2} - \ldots - \sigma_1(a, b)\right].$$

Proof. From Lemma 3.1.2 we see that H/H' is a free abelian group freely generated by $aH', a^bH', \ldots, a^{b^{r-1}}H'$. Now with respect to this basis the action of b on H/H' can be expressed by the matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \sigma_1(a, b) & \sigma_2(a, b) & \sigma_3(a, b) & \cdots & \sigma_{r-1}(a, b) & \sigma_r(a, b) \end{bmatrix}$$

which has the required characteristic polynomial. //

3.1.5 THEOREM. *Let* C *and* E *be isomorphic groups in* $X(r)$ *and* θ *the isomorphism of* C *onto* E *. Assume that* C/C' *is not free abelian of rank* 2 *.*

Let (c, d) *and* (e, f) *be generating pairs associated with standard presentations of* C *and* E *respectively. Then,* $(d\theta)f^{-\eta}$ *belongs to* $\langle e\rangle^E$ *where* $\eta = \pm 1$ *, and* $\sigma_k(c, d) = \sigma_k(c, d^\eta)$ *for* $k = 1, 2, \ldots, r$ *.*

Proof. Since C and E are isomorphic so are C/C' and E/E' . Thus both C/C' and E/E' are the direct product of an infinite cyclic group and a cyclic group of order s , where s is some positive integer.

Put $M = \langle c\rangle^C$ and $N = \langle e\rangle^E$.

Now $C/M \cong C\theta/M\theta = E/M\theta$ and C/M is an infinite cyclic group. From Lemma 3.1.3 we conclude that $M\theta = N$. It follows that θ induces an isomorphism between the infinite cyclic groups C/M and E/N so clearly $(d\theta)f^{-\eta}$ belongs to $\langle e\rangle^E$ where $\eta = 1$ or $\eta = -1$.

Also, θ induces an isomorphism θ^* of M/M' onto N/N' such that if g belongs to M then

$$
\begin{aligned}
[(gM')^d]\theta^* &= (g^d M')\theta^* \\
&= (g^d\theta)N' \\
&= (g\theta)^{d\theta}N' \\
&= (g\theta)^{f^n h}N' \quad \text{where } h \text{ belongs to } N\,, \\
&= (g\theta)^{f^n}N' \\
&= [(g\theta)N']^{f^n}\,.
\end{aligned}
$$

Now θ is an isomorphism so it follows that the characteristic polynomials of the action of d on M/M' and of f^n on N/N' are the same. Lemma 3.1.4 now yields the required results. //

3.2 Cyclic extensions of free groups of rank 2

In this section we characterize the groups in $X(2)$ by using Theorem 3.1.5. The results for groups in $X(2)$ whose commutator quotient group is free of rank one have already been obtained in Theorem 1 of [19].

3.2.1 LEMMA. *The group* $C = \mathrm{gp}\left[c,\, d;\, c^{d^2} = c\right]$ *and the group* $E = \mathrm{gp}\left[e,\, f;\, e^{f^2} = e^{-1}e^{2f}\right]$ *are not isomorphic.*

Proof. Let V and W be the verbal subgroups of C and E generated by the third powers of the elements in C and E respectively. Then firstly, C/V is the direct product of two 3-cycles. Also the relation $e^{f^2} = e^{-1}e^{2f}$ can be rewritten in the form $fef^{-2}ef = e^2$, or, $(fe)^3(f^{-3})^{efe}e^{-3} = 1$. It follows that E/W is the absolutely free two generator group of exponent 3 and is a non-abelian group of order 27 (see [5], p. 80]). Thus C and E cannot be isomorphic. //

3.2.2 THEOREM. *The generating pairs of a group in* $\mathfrak{X}(2)$ *associated with a one relator presentation lie in the same A-class.*

Proof. Let G be a group in $\mathfrak{X}(2)$. Then G has a standard presentation (by Theorem 3.1.1) of the form

$G = \mathrm{gp}\left(a,\ b;\ a^{b^2} = a^{kb}a^{\varepsilon}a^{lb}\right)$, where k and l are integers and $\varepsilon = \pm 1$.

Now the relation $a^{b^2} = a^{kb}a^{\varepsilon}a^{lb}$ can be written in the form $a^{-kb}a^{b^2}a^{kb} = a^{\varepsilon}a^{(l+k)b}$. But $a^{-kb}a^{b^2}a^{kb} = a^{b^2a^{kb}} = a^{a^kba^kb}$.
Putting $d = a^kb$ we see that $G = \mathrm{gp}\left(a,\ d;\ a^{d^2} = a^{\varepsilon}a^{(l+k)d}\right)$.
Further $(a,\ a^kb)$ is clearly in the same A-class as $(a,\ b)$. It follows that we need only consider standard presentations of the form $\left(x,\ y;\ x^{-y^2}x^{\varepsilon}x^{ry}\right)$ for some integer r and $\varepsilon = \pm 1$.

We determine conditions under which two such standard presentations define isomorphic groups. Thus let $C = \mathrm{gp}\left(c,\ d;\ c^{d^2} = c^{\varepsilon_1}c^{r_1d}\right)$ and $E = \mathrm{gp}\left(e,\ f;\ e^{f^2} = e^{\varepsilon_2}e^{r_2f}\right)$ be isomorphic groups, where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$ and r_1, r_2 are integers. Clearly, since C/C' and E/E' are isomorphic $r_1 + \varepsilon_1 - 1 = \pm(r_2+\varepsilon_2-1)$.

We dispense first with the case in which $r_1 + \varepsilon_1 - 1 = r_2 + \varepsilon_2 - 1 = 0$. When $\varepsilon_1 = 1$ we have $r_1 = 0$ and when $\varepsilon_1 = -1$ we have $r_1 = 2$. Thus we have the groups $C = \mathrm{gp}\left(c,\ d;\ c^{d^2} = c\right)$ and $E = \mathrm{gp}\left(e,\ f;\ e^{f^2} = e^{-1}e^{f}\right)$ to consider, and by Lemma 3.2.1 these groups are not isomorphic.

Suppose now that $r_1 + \varepsilon_1 - 1 = \pm(r_2+\varepsilon_2-1) \neq 0$. By Theorem 3.1.5 we have two cases to consider;

(a) $\varepsilon_1 = \sigma_1(c, d) = \sigma_1(e, f) = \varepsilon_2$ and

$$r_1 = \sigma_2(c, d) = \sigma_2(e, f) = r_2 \text{ , and}$$

(b) $\varepsilon_1 = \sigma_1(c, d) = \sigma_1\left(e, f^{-1}\right) = \varepsilon_2$ and

$$r_1 = \sigma_2(c, d) = \sigma_2\left(e, f^{-1}\right) = -r_2\varepsilon_2 \text{ .}$$

Suppose that case (b) holds.

Firstly if $r_1 + \varepsilon_1 - 1 = r_2 + \varepsilon_2 - 1$ then as $\varepsilon_1 = \varepsilon_2$ we have $r_1 = r_2$. We are left with $r_1 + \varepsilon_1 - 1 = -r_2 - \varepsilon_2 + 1$, $\varepsilon_1 = \varepsilon_2$ and $r_1 = -r_2\varepsilon_2$. If $\varepsilon_1 = \varepsilon_2 = -1$ then $r_1 = r_2$. When $\varepsilon_1 = \varepsilon_2 = 1$ then $r_1 = -r_2$.

It follows that if C and E are isomorphic groups then $\varepsilon_1 = \varepsilon_2$ and $r_1 = r_2$ unless $\varepsilon_1 = \varepsilon_2 = 1$ and $r_1 = -r_2$. However, when $\varepsilon_2 = 1$ we have associated with the generating pair $\left(e, f^{-1}\right)$ of E the presentation $\left\{x, y; x^{-y^2}xx^{-r_2y}\right\}$ which is also the presentation associated with the generating pair (c, d) of C when $r_1 = -r_2$ and $\varepsilon_1 = 1$. Also $\left(e, f^{-1}\right)$ is in the same A-class as (e, f).

To summarize then a generating pair which is associated with a standard presentation of a group G in $\mathrm{X}(2)$ lies in the same A-class as a generating pair associated with a presentation $\left\{x, y; x^{-y^2}x^{\varepsilon}x^{ry}\right\}$. These presentations for different ε and r define non-isomorphic groups unless $\varepsilon = 1$ when the generating pair (a, b) associated with the presentation $\left\{x, y; x^{-y^2}xx^{ry}\right\}$ and the generating pair $\left(a, b^{-1}\right)$ associated with $\left\{x, y; x^{-y^2}xx^{-ry}\right\}$ lie in the same A-class and these two presentations define isomorphic groups.

We conclude that the generating pairs of a group in $X(2)$ associated with one relator presentations lie in the same A-class. //

We have proved rather more in the theorem than stated and we put this in the following corollary.

3.2.3 COROLLARY. *Let G be a group in $X(2)$ and let s be the torsion number of G.*

If $s = 0$ then G is one of the two non-isomorphic groups with presentation $\{x, y; x^{-y^2}x\}$ *or* $\{x, y; x^{-y^2}x^{-1}x^{2y}\}$.

If $s \neq 0$ then G is one of the three non-isomorphic groups with presentation $\{x, y; x^{-y^2}xx^{sy}\}$, $\{x, y; x^{-y^2}x^{-1}x^{(2+s)y}\}$ *or* $\{x, y; x^{-y^2}x^{-1}x^{(2-s)y}\}$.

3.3 The second isomorphism criteria

As was shown in §3.2 the results of §3.1 are sufficient to determine the groups in $X(2)$. Also, as an example, Theorem 3.1.5 can be used to show that the presentations $\{x, y; x^{-y^3}xx^{2y}\}$ and $\{x, y; x^{-y^3}xx^{y}x^{y^2}\}$ define non-isomorphic groups. However, we are powerless to decide, using Theorem 3.1.5, whether the presentations $\{x, y; x^{-y^3}xx^{y}x^{y^2}\}$ and $\{x, y; x^{-y^3}xx^{y^2}x^{y}\}$ define isomorphic groups or not. The results of this section will enable us to show in §3.4 that the groups defined by these presentations are not isomorphic.

Since the groups in $X(2)$ were dealt with in §3.2 we will always consider a class $X(r)$ with r greater than 2 in this section.

Let G belong to $X(r)$ and suppose that G has a standard presentation (3.1). Let s denote the torsion number of G. Throughout this section we restrict ourselves to groups G in $X(r)$ with G/G' not torsion free. In other words we suppose that $s \neq 0, 1$.

We define an $(r-1)(s-1)$ tuple called a *standard vector* which will be used in investigating the isomorphism problem.

Let

$$\alpha_k = a^{sb^{k-1}} \quad (1 \leq k \leq r) ,$$

and

$$\alpha_{i,j} = \left(a^{-b^{j-1}} a^{b^j}\right)^{a^{i-1}} \quad (1 \leq i \leq s-1,\ 1 \leq j \leq r-1) .$$

It will be shown in Lemma 3.3.2 that the set

$$\{\alpha_k,\ \alpha_{i,j}\ ;\ 1 \leq k \leq r,\ 1 \leq i \leq s-1,\ 1 \leq j \leq r-1\}$$

constitutes a free basis for G'. Consequently, the set $\{\alpha_k G'',\ \alpha_{i,j} G'';\ 1 \leq k \leq r,\ 1 \leq i \leq s-1,\ 1 \leq j \leq r-1\}$ constitutes a free basis for the abelian group G'/G'' (which we write additively).

Since $a^{-b^{r-1}} a^{b^r}$ belongs to G' there exist uniquely determined integers u_k $(1 \leq k \leq r)$ and $u_{i,j}$ $(1 \leq i \leq s-1,\ 1 \leq j \leq r-1)$ such that

$$a^{-b^{r-1}} a^{b^r} G'' = \sum_{k=1}^{r} u_k \alpha_k + \sum_{j=1}^{r-1} \sum_{i=1}^{s-1} u_{i,j} \alpha_{i,j} . \tag{3.3}$$

The $(r-1)(s-1)$ tuple

$$u = \left(u_{1,1},\ u_{2,1},\ \ldots,\ u_{s-1,1},\ \ldots,\ u_{1,r-1},\ u_{2,r-1},\ \ldots,\ u_{s-1,r-1}\right)$$

is called a *standard vector* obtained from the given standard presentation of G.

For example, let

$$G = \mathrm{gp}\left(a,\ b;\ a^{b^4} = a^{b^2}a^{-b^3}a^{2b}aa^{2b^3}a^{-b}\right).$$

Here $r = 4$ and $s = 3$.

Now a basis for G' consists of the elements a^3, a^{3b}, a^{3b^2}, a^{3b^3}, $a^{-1}a^b$, $(a^{-1}a^b)^a$, $a^{-b}a^{b^2}$, $\left(a^{-b}a^{b^2}\right)^a$, $a^{-b^2}a^{b^3}$, $\left(a^{-b^2}a^{b^3}\right)^a$. Thus we may write

$$a^{-b^3}a^{b^4} = a^{-b^3}a^{b^2}a^{-b^3}a^{2b}aa^{2b^3}a^{-b} = \left(a^{-b^3}a^{b^2}\right)\left(a^{-b^3}a^{b^2}\right)\left(a^{-b^2}a^{b}\right)(a^{3b})$$

$$(a^{-b}a)(a^{-1}a^b)^a(a^{-1}a^b)(a^{3b})(a^{-b}a)\left(a^{-b^3}a^{b^2}\right)^a(a^{-1}a^b).$$

Consequently, $u_2 = 2$, $u_{2,1} = 1$, $u_{1,2} = -1$, $u_{2,2} = -1$, $u_{1,3} = -2$ and $u_{2,3} = -1$ so that $\mathrm{u} = (0, 1, -1, -1, -2, -1)$. //

Notation. Let u be a standard vector obtained from a standard presentation (3.1), and s the torsion number of a group G in $X(r)$. We put $u_{0,j} = 0$ for $j = 1, 2, \ldots, r-1$, and if i is any integer then we define $u_{i,j}$ to be $u_{i',j}$, where i' is an integer given by $0 \le i' \le s-1$ and $i \equiv i'$ modulo s.

Remark. The integers $\sigma_k(a, b)$ for $k = 1, 2, \ldots, r$ were defined in §3.2. It can be shown that

$$u_1 = 1/s\left[\sigma_1(a, b) + \sum_{i=1}^{s-1} u_{i,1}\right]$$

$$u_2 = 1/s\left[\sigma_2(a, b) - \left(\sum_{i=1}^{s-1} u_{i,1} - \sum_{i=1}^{s-1} u_{i,2}\right)\right]$$

$$\cdots\cdots\cdots\cdots$$

$$u_{r-1} = 1/s\left[\sigma_{r-1}(a, b) - \left(\sum_{i=1}^{s-1} u_{i,r-2} - \sum_{i=1}^{s-1} u_{i,r-1}\right)\right]$$

$$u_r = 1/s\left[\sigma_r(a, b) - \sum_{i=1}^{s-1} u_{i,r-1}\right].$$

It follows from this that the integers $u_1, \ldots, u_r$ are determined by the integers $\sigma_1(a, b), \ldots, \sigma_r(a, b)$ and u . //

Let C and E belong to $\mathrm{X}(r)$ and suppose that θ is an isomorphism of C onto E . Then it is clear, since C/C' and E/E' are isomorphic, that the torsion numbers of C and of E are equal. Let s be this torsion number and suppose that $s \neq 0, 1$.

Let $C = \mathrm{gp}\left(c, d;\ c^{d^r} = wc^{\varepsilon}w'\right)$ and $E = \mathrm{gp}\left(e, f;\ e^{f^r} = ve^{\varepsilon'}v'\right)$ be standard presentations. By Theorem 3.1.5 we have that $d\theta\langle e\rangle^E = f^{\eta}\langle e\rangle^E$, where $\eta = \pm 1$. However, since by (3.2) the group E when presented with generating pair $\left(e, f^{-1}\right)$ is also a standard presentation, we may assume without loss in generality that $d\theta\langle e\rangle^E = f\langle e\rangle^E$. In this case we say that the standard presentations of C and E are *θ-compatible.*

The following theorem, which is the main theorem of this section, relates the standard vectors of isomorphic groups in $\mathrm{X}(r)$ whose commutator quotient groups are not torsion free.

3.3.1 THEOREM. *Let θ be an isomorphism between two groups C and E in $\mathrm{X}(r)$. Let s be the torsion number of C and E and suppose that $s \neq 0, 1$. Let w and v be standard vectors belonging to θ-compatible standard presentations of C and E .*

Then for each $j = 1, 2, \ldots, r-1$ there is a permutation $\theta(j)$ of the set $\{0, 1, \ldots, s-1\}$ such that

$$w_{p,j} - v_{p\theta(j),j} = 1/s\left(\sum_{i=1}^{s-1} w_{i,j} - \sum_{i=1}^{s-1} v_{i,j}\right)$$

for each p in $\{0, 1, \ldots, s-1\}$.

Moreover, $\theta(j)$ is the particular mapping of $\{0, 1, \ldots, s-1\}$ onto itself given by $p\theta(j) = kp - [(k-1)+(r-j)m]$, for some integers

k *and* m *with* $1 \le k \le s-1$, $(k, s) = 1$, *and* $0 \le m \le s-1$. //

Before proving Theorem 3.3.1 we have to establish a number of preliminary results.

Let G belong to $\mathfrak{X}(r)$. We will consider a particular quotient group of G'/G'' and the action of elements of G on it.

First we find a free basis for G' .

3.3.2 LEMMA. *Let* G *be a group in* $\mathfrak{X}(r)$ *with a standard presentation* $G = \mathrm{gp}\left(a, b;\, a^{b^r} = ua^{\varepsilon}u'\right)$. *Suppose that* s *the torsion number of* G *is neither* 0 *nor* 1 . *Then a set of free generators for* G' *is given by the elements*

$$\alpha_k = a^{sb^{k-1}} \quad (1 \le k \le r) ,$$

and

$$\alpha_{i,j} = \left(a^{-b^{j-1}}a^{b^j}\right)^{a^{i-1}} \quad (1 \le i \le s-1,\ 1 \le j \le r-1) .$$

Proof. By Lemma 3.1.2 we have that $\langle a\rangle^G$ is a free group of rank r generated by $a, a^b, \ldots, a^{b^{r-1}}$. We choose $\{1, a, \ldots, a^{s-1}\}$ as a set of coset representatives for $\langle a\rangle^G$ modulo G' .

If v is an element of $\langle a\rangle^G$ then let $\overline{v}$ denote the representative of the coset vG' in the set $\{1, a, \ldots, a^{s-1}\}$.

By Theorem 2.7 of [12] we see that the non-trivial elements $\tau(\alpha, \beta) = \alpha\beta\cdot\overline{\alpha\beta}^{\,-1}$, where α belongs to $\{1, a, \ldots, a^{s-1}\}$ and β belongs to $\left\{a, a^b, \ldots, a^{b^{r-1}}\right\}$, generate G' .

But for $0 \le i \le s-1$ and $0 \le j \le r-1$ we have

$$\tau\left[a^{i}, a^{b^{j}}\right] = a^{i}a^{b^{j}}.\overline{a^{i}a^{b^{j}}}^{-1}$$

$$= a^{i}a^{b^{j}}.\overline{a^{-(i+1)}} ,$$

so that the elements $a^{s}.a^{-1}a^{b^{j}}$ $(0 \le j \le r-1)$ and $\left[a^{-1}a^{b^{j}}\right]^{a^{-(i+1)}}$ $(1 \le j \le r-1,\ 0 \le i \le s-2)$ generate G'. Now for $j = 0$ we have $a^{s}.a^{-1}a^{b^{j}} = a^{s}$ so that equivalently G' is generated by the elements a^{s} and $\left(a^{-1}a^{b^{j}}\right)^{a^{-i}}$ $(1 \le j \le r-1,\ 0 \le i \le s-1)$.

Now G' is a free group of rank $s(r-1) + 1$, as can be computed by the Schreier formula (Theorem 2.10 [12]). Since the cardinality of the set $\{\alpha_{k}, \alpha_{i,j};\ 1 \le k \le r,\ 1 \le j \le r-1,\ 1 \le i \le s-1\}$ is also $s(r-1) + 1$ if these elements generate G' then they generate G' freely.

Let N be the subgroup of G' generated by $\{\alpha_{k}, \alpha_{i,j};\ 1 \le k \le r,\ 1 \le j \le r-1,\ 1 \le i \le s-1\}$. Since $\alpha_{1} = a^{s}$ it suffices to show that $\left(a^{-1}a^{b^{j}}\right)^{a^{i}}$ belongs to N for each i and j with $0 \le i \le s-1$ and $1 \le j \le r-1$.

Firstly,

$$a^{-1}a^{b^{j}} = a^{-1}a^{b}.a^{-b}a^{b^{2}}\ .\ \ldots\ .\ a^{-b^{j-1}}a^{b^{j}}$$

$$= \alpha_{1,1}\alpha_{1,2}\ \ldots\ \alpha_{1,j}$$

so that $a^{-1}a^{b^{j}}$ belongs to N. We show that a normalizes N in order to complete the proof.

But

$$\alpha_k^a = a^{-1}a^b . a^{-b}a^{b^2} . \ldots . a^{-b^{j-2}}a^{b^{j-1}} . a^{sb^{j-1}} . a^{-b^{j-1}}a^{b^{j-2}} . \ldots . a^{-b^2}a^b . a^{-b}a$$

$$= \alpha_{1,1}\alpha_{1,2} \ldots \alpha_{1,j-1}\alpha_j\alpha_{1,j-1}^{-1} \ldots \alpha_{1,2}^{-1}\alpha_{1,1}^{-1} ,$$

$\alpha_{i,j}^a = \alpha_{i+1,j}$ when $i = 1, 2, \ldots, s-1$,

and $\alpha_{s-1,j}^a = \alpha_1^{-1}\alpha_{1,j}\alpha_1$. //

Our next lemma will require the concept of transfer for which we refer to [21] (p. 60, §3.5), where proofs and statements of standard properties are given. The use of the transfer here is due to Dr R.M. Bryant and replaces a previously more combinatorial argument.

Let K be a group and N a subgroup of finite index n in K . Let $\{t_1, \ldots, t_n\}$ be a set of left coset representatives of N in K , and let g be an arbitrary element of K . Then for each $i = 1, 2, \ldots, n$ there is a j with $j = 1, 2, \ldots, n$ and an h in N such that $gt_i = t_jh$. The transfer T of K into N/N' is defined by setting

$$gT = \prod_{i=1}^{n} \left(t_j^{-1}gt_i\right)N' .$$

The transfer is independent of the choice of left coset representatives $\{t_1, \ldots, t_n\}$ and depends only on K and N .

3.3.3 LEMMA. *Let W be the subgroup of G'/G'' generated by $\left\{a^{sb^{k-1}}G''; 1 \le k \le r\right\}$ and let $H = \langle a\rangle^G$. Then W is a G-invariant subgroup of G'/G'' and there is a G-isomorphism ξ of H/H' onto W such that $\left(a^{b^{k-1}}H'\right)\xi = a^{sb^{k-1}}G''$ for $k = 1, 2, \ldots, r$.*

Proof. Let T be the transfer of H into G'/G'' . As the image of H under T is abelian H' is contained in the kernel of T. Thus T defines a homomorphism ξ of H/H' into G'/G''.

Suppose that c is an arbitrary element of G and take $\{1, a^c, a^{2c}, \ldots, a^{(s-1)c}\}$ as a set of left coset representatives of G' in H. Then,

$$(a^c)T = [(a^{-c}.a^c.1).(a^{-2c}.a^c.a^c) \ldots (a^{-(s-1)c}.a^c.a^{(s-1)c}).(1.a^c.a^{(s-1)c})]G''$$

$$= a^{sc}G''.$$

By Lemma 3.3.2 we have that W is a free abelian subgroup of G'/G''. Since by Lemma 3.1.2 the set of elements $\left\{a^{b^{k-1}}H';\ 1 \le k \le r\right\}$ generates H/H' freely and $\left(a^{b^{k-1}}H'\right)\xi = a^{sb^{k-1}}G''$ we see that ξ is an isomorphism of H/H' onto W.

It remains to show that ξ commutes with the action of elements of G, and for this it is enough to show that T commutes with the action of elements of G.

But if c and d are arbitrary elements of G then

$$\begin{aligned}[(a^c)T]^d &= [a^{sc}G'']^d \\ &= a^{scd}G'' \\ &= [a^{cd}G'']T \\ &= [(a^cG'')^d]T. \quad //\end{aligned}$$

Let $V = (G'/G'')/W$, where W is defined in Lemma 3.3.3. As W is G-invariant the elements a and b of G induce automorphisms of V whose action is given in the following Lemma 3.3.4.

First however we define $\varepsilon_{i,j}$ to be $(\alpha_{i,j}G'')W$ for i and j with $1 \le i \le s-1$ and $1 \le j \le r-1$. It is clear that the set

$\{\varepsilon_{i,j};\ 1 \le i \le s\text{-}1,\ 1 \le j \le r\text{-}1\}$ constitutes a free basis for V.

3.3.4 LEMMA. *(i) The action of* a *on the basis* $\{\varepsilon_{i,j};\ 1 \le i \le s\text{-}1,\ 1 \le j \le r\text{-}1\}$ *of* V *is given by*

$$\varepsilon_{i,j}^{a} = \begin{cases} \varepsilon_{i+1,j} & \text{if } 1 \le i < s-1\,, \\ -\sum\limits_{k=1}^{s-1} \varepsilon_{k,j} & \text{if } i = s-1\,, \end{cases}$$

for $j = 1, 2, \ldots, r\text{-}1$.

(ii) The action of b *on the basis* $\{\varepsilon_{i,j};\ 1 \le i \le s\text{-}1,\ 1 \le j \le r\text{-}1\}$ *of* V *is given by*

$$\varepsilon_{i,j}^{b} = \begin{cases} \varepsilon_{i,j+1} & \text{if } 1 \le j < r-1\,, \\ \sum\limits_{l=1}^{r-1} \sum\limits_{k=1}^{s-1} u_{k,l}\, \varepsilon_{k,l}^{a^{i-1}} & \text{if } j = r-1\,, \end{cases}$$

for $i = 1, 2, \ldots, s\text{-}1$.

Proof. We remark that if n is any integer, and z is any element of G', then $z^{a^{nb^j}} G'' = z^{a^n \cdot a^{-n} a^{nb^j}} G'' = z^{a^n} G''$.

(i) Suppose that $1 \le i \le s\text{-}1$. Then

$$\varepsilon_{i,j}^{a} = \left[\left(a^{-b^{j-1}} a^{b^j}\right)^{a^{i-1}} G''W\right]^{a}$$

$$= \left(a^{-b^{j-1}} a^{b^j}\right)^{a^{i}} G''W\,.$$

Thus if $1 \le i < s\text{-}1$ then $\varepsilon_{i,j}^{a} = \varepsilon_{i+1,j}$.

Also $\varepsilon_{s-1,j}^{a} = \left(a^{-b^{j-1}} a^{b^j}\right)^{a^{s-1}} G''W$. But

$$\left(a^{-b^{j-1}}a^{b^j}\right)^{a^{s-1}} = a^{-(s-1)}a^{(s-1)b^{j-1}} . a^{-sb^{j-1}}a^{sb^j} .$$

$$. a^{-(s-1)b^j}a^{(s-1)b^{j-1}} . a^{-(s-1)b^{j-1}}a^{s-1} ,$$

hence

$$\varepsilon^a_{s-1,j} = a^{-(s-1)b^j}a^{(s-1)b^{j-1}}G''W .$$

Now

$$a^{-(s-1)b^{j-1}}a^{(s-1)b^j} = a^{-b^{j-1}}a^{b^j} . \left(a^{-b^{j-1}}a^{b^j}\right)^{a^{b^j}} . \ldots .$$

$$\left(a^{-b^{j-1}}a^{b^j}\right)^{a^{(s-2)b^j}} ,$$

so that

$$\varepsilon^a_{s-1,j} = - \sum_{k=1}^{s-1} \varepsilon_{k,j} .$$

(ii) Suppose that $1 \le j \le r-1$. Then

$$\varepsilon^b_{i,j} = \left(a^{-b^j}a^{b^{j+1}}\right)^{a^{(i-1)b}} G''W$$

$$= \left(a^{-b^j}a^{b^{j+1}}\right)^{a^{i-1}} G''W$$

using our remark above. Thus if $1 \le i < r-1$ then $\varepsilon^b_{i,j} = \varepsilon_{i,j+1}$.

Also

$$\varepsilon^b_{i,r-1} = \left(a^{-b^{r-1}}a^{b^r}\right)^{a^{i-1}} G''W$$

$$= \sum_{l=1}^{r-1}\sum_{k=1}^{s-1} u_{k,l}\varepsilon^{a^{i-1}}_{k,l}$$

from (3.3). //

Now V is a free abelian group which we consider as a free

Z-module having $\{\varepsilon_{i,j};\ 1 \leq i \leq s-1,\ 1 \leq j \leq r-1\}$ as a free basis.
By a standard process, to which we refer to §11 of [3], V can be extended to a C-module by tensoring; $C \otimes_Z V$, where C is regarded as a Z-module.

If L is any Z-linear mapping from V into V then L extends to a C-linear mapping $I \otimes L$ from $C \otimes V$ into $C \otimes V$, where I is the identity mapping of C.

Now it is clear that $\{1 \otimes v;\ v \in V\}$ not only forms a module isomorphic to V (with which we identify V) but, since $\{\varepsilon_{i,j};\ 1 \leq i \leq s-1,\ 1 \leq j \leq r-1\}$ constitutes a basis for V, we see that $\{1 \otimes \varepsilon_{i,j};\ 1 \leq i \leq s-1,\ 1 \leq j \leq r-1\}$ constitutes a basis for $C \otimes V$. Thus if v is any element of $C \otimes V$ we may write v as $\sum_{i,j} \nu_{i,j}(1 \otimes \varepsilon_{i,j})$ where each $\nu_{i,j}$ belongs to C and, after making the identification of $1 \otimes \varepsilon_{i,j}$ with $\varepsilon_{i,j}$, as an element $\sum_{i,j} \nu_{i,j}\varepsilon_{i,j}$. In this way we may regard $C \otimes V$ as consisting of all elements $\sum_{k} \mu_k v_k$, where each v_k belongs to V and μ_k belongs to C.

I am indebted to Dr R.M. Bryant for suggesting the tensoring of C with V here. It results in a more conceptual and less complicated proof than an earlier proof. The advantage of dealing with $C \otimes V$ is that $C \otimes V$ can be decomposed into certain G-invariant submodules, as we show in Lemma 3.3.6. We then calculate in Lemma 3.3.8 the characteristic polynomials of the action of the elements ba^t for $t = 0, 1, \ldots, s-1$ on these G-invariant submodules. In Lemma 3.3.10 we show that in isomorphic groups the characteristic polynomials just calculated correspond in an easily identifiable way. This will then provide the information needed to prove Theorem 3.3.1.

Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ denote the distinct non-trivial s-th roots of unity and let Λ denote the matrix

$$\begin{bmatrix} \lambda_1^{s-1}-1 & \lambda_1^{s-2}-1 & \ldots & \lambda_1 & -1 \\ \lambda_2^{s-1}-1 & \lambda_2^{s-2}-1 & \ldots & \lambda_2 & -1 \\ \cdot & \cdot & \ldots & \cdot & \\ \lambda_{s-1}^{s-1}-1 & \lambda_{s-1}^{s-2}-1 & \ldots & \lambda_{s-1} & -1 \end{bmatrix} .$$

Then Λ is a non-singular matrix since it has an inverse given by the matrix

$$\begin{bmatrix} \lambda_1/s & \lambda_2/s & \ldots & \lambda_{s-1}/s \\ \lambda_1^2/s & \lambda_2^2/s & \ldots & \lambda_{s-1}^2/s \\ \cdot & \cdot & \ldots & \cdot \\ \lambda_1^{s-1}/s & \lambda_2^{s-1}/s & \ldots & \lambda_{s-1}^{s-1}/s \end{bmatrix}$$

as can be seen from the following lemma.

3.3.5 LEMMA. *The expression*

$$\left(\lambda_i^{s-1}-1\right)\left(\lambda_k/s\right) + \left(\lambda_i^{s-2}-1\right)\left(\lambda_k^2/s\right) + \ldots + \left(\lambda_i-1\right)\left(\lambda_k^{s-1}/s\right)$$

is 0 *if* $k \neq i$ *and* 1 *if* $k = i$, *where* k *and* i *are integers with* $1 \leq k, i \leq s$.

Proof. Firstly

$$\left(\lambda_i^{s-1}-1\right)\left(\lambda_k/s\right) + \ldots + \left(\lambda_i-1\right)\left(\lambda_k^{s-1}/s\right)$$

$$= (1/s)\Big[\left(\lambda_k/\lambda_i\right) + \left(\lambda_k/\lambda_i\right)^2 + \ldots$$

$$\ldots + \left(\lambda_k/\lambda_i\right)^{s-1} - \lambda_k - \lambda_k^2 - \ldots - \lambda_k^{s-1}\Big] .$$

Now if $k \neq i$, as λ_k/λ_i is a non-trivial s-th root of unity, we have $(\lambda_k/\lambda_i) + (\lambda_k/\lambda_i)^2 + \ldots + (\lambda_k/\lambda_i)^{s-1} = -1$. Since $1 + \lambda_k + \lambda_k^2 + \ldots + \lambda_k^{s-1} = 0$ we have the result in this case.

Suppose that $k = i$. Then $(\lambda_k/\lambda_i) + \ldots + (\lambda_k/\lambda_i)^{s-1} = -1$. and since $-\lambda_k - \lambda_k^2 - \ldots - \lambda_k^{s-1} = 1$ we have the result in this case as well. //

Now for each i with $1 \leq i \leq s-1$ and each j with $1 \leq j \leq r-1$ set

$$\beta_{i,j} = \left(\lambda_i^{s-1}-1\right)\varepsilon_{1,j} + \left(\lambda_i^{s-2}-1\right)\varepsilon_{2,j} + \ldots + (\lambda_i-1)\varepsilon_{s-1,j} .$$

Since $\{\varepsilon_{i,j};\ 1 \leq i \leq s-1,\ 1 \leq j \leq r-1\}$ constitutes a free basis for $C \otimes V$ and Λ is invertible the set $\{\beta_{i,j};\ 1 \leq i \leq s-1,\ 1 \leq j \leq r-1\}$ constitutes a free basis for $C \otimes V$.

3.3.6 LEMMA. *For each integer i with $1 \leq i \leq s-1$ let V_i be the submodule of $C \otimes V$ generated by $\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,r-1}$. Then V_i is an eigenspace of $C \otimes V$ under the action of a corresponding to the eigenvalue λ_i. Further, each V_i is a G-invariant submodule of $C \otimes V$ and $C \otimes V = V_1 \oplus \ldots \oplus V_{s-1}$.*

Proof. Using Lemma 3.3.4 *(i)* we have

$$\beta_{i,j}^{a} = \left(\lambda_i^{s-1}-1\right)\varepsilon_{2,j} + \left(\lambda_i^{s-2}-1\right)\varepsilon_{3,j} + \cdots$$

$$\cdots + \left(\lambda_i^{2}-1\right)\varepsilon_{1,j} + \left(\lambda_i-1\right)\left(-\sum_{i=1}^{s-1}\varepsilon_{i,j}\right)$$

$$= -\left(\lambda_i-1\right)\varepsilon_{1,j} + \left(\left(\lambda_i^{s-1}-1\right)-\left(\lambda_i-1\right)\right)\varepsilon_{2,j} + \cdots$$

$$\cdots + \left(\left(\lambda_i^{2}-1\right)-\left(\lambda_i-1\right)\right)\varepsilon_{s-1,j}$$

$$= \lambda_i\left(\lambda_i^{s-1}-1\right)\varepsilon_{1,j} + \lambda_i\left(\lambda_i^{s-2}-1\right)\varepsilon_{2,j} + \cdots + \lambda_i\left(\lambda_i-1\right)\varepsilon_{s-1,j}$$

$$= \lambda_i\beta_{i,j} ,$$

where $1 \le i \le s-1$ and $1 \le j \le r-1$.

Suppose that v belongs to $\mathsf{C} \otimes V$ and that $v = \sum_{j=1}^{r-1}\sum_{k=1}^{s-1} v_{k,j}\beta_{k,j}$, where each $v_{k,j}$ belongs to C. If $v^a = \lambda_i v$, for some i with $1 \le i \le s-1$, then

$$\sum_{j=1}^{r-1}\sum_{k=1}^{s-1} v_{k,j}\lambda_k\beta_{k,j} = \lambda_i \sum_{j=1}^{r-1}\sum_{k=1}^{s-1} v_{k,j}\beta_{k,j}$$

so that $\sum_{j=1}^{r-1}\sum_{k=1}^{s-1} \left(\lambda_i-\lambda_k\right)v_{k,j}\beta_{k,j} = 0$. Since $\lambda_i - \lambda_k \ne 0$ for $i \ne k$ we must have $v_{k,j} = 0$. Thus $v = \sum_{j=1}^{r-1} v_{i,j}\beta_{i,j}$ and v belongs to V_i. It follows that V_i is an eigenspace of $\mathsf{C} \otimes V$ under the action of a and corresponds to the eigenvalue λ_i.

If v belongs to V_i then $\left(v^b\right)^a = v^{ba} = v^{ab} = \left(\lambda_i v\right)^b = \lambda_i v^b$ so that v^b also belongs to V_i. It follows that V_i is G-invariant. The last statement in the lemma follows because the set $\{\beta_{i,j};\ 1 \le i \le s-1,\ 1 \le j \le r-1\}$ generates $\mathsf{C} \otimes V$ freely. //

3.3.7 LEMMA. *Let W be the subgroup of G'/G'' generated by* *the set* $\left\{a^{sb^{k-1}}G''; 1 \le k \le r\right\}$. *Then* W *is the set of elements in* G'/G'' *fixed under the action of the elements of* $\langle a\rangle^G$.

Proof. We remark that if g is an element of G'/G'' then if j is any integer $g^{a^{b^j}}G'' = g^{a \cdot a^{-1}a^{b^j}}G'' = g^a G''$. It follows that we need only consider the set of elements in G'/G'' fixed under the action of a.

Now W is fixed under the action of a since

$$\left(a^{sb^{k-1}}G''\right)^a = a^{-1}a^{b^{k-1}} \cdot a^{sb^{k-1}} \cdot a^{-b^{k-1}} aG'' = a^{sb^{k-1}}G''$$

for each $k = 1, 2, \ldots, r-1$. But by Lemma 3.3.6 we have $\mathbb{C} \otimes V = V_1 \oplus \cdots \oplus V_{s-1}$ and each V_i is an eigenspace of $\mathbb{C} \otimes V$ corresponding to a non-trivial eigenvalue λ_i of a for $i = 1, 2, \ldots, s-1$. Thus no element of $V = (G'/G'')/W$ is fixed under the action of a and we may conclude that W actually equals the set of elements of G'/G'' fixed under the action of a. //

It follows from (3.3) that

$$a^{-b^{r-1}}a^{b^r}G''W = \sum_{j=1}^{r-1}\sum_{i=1}^{s-1} u_{i,j}\varepsilon_{i,j}.$$

Since

$$\beta_{i,j} = \left(\lambda_i^{s-1}-1\right)\varepsilon_{1,j} + \cdots + \left(\lambda_i-1\right)\varepsilon_{s-1,j}$$

we see that

$$\sum_{j=1}^{r-1}\sum_{i=1}^{s-1} u_{i,j}\varepsilon_{i,j} = \sum_{j=1}^{r-1}\sum_{i=1}^{s-1} c_{i,j}\beta_{i,j}$$

for some elements $c_{i,j}$ belonging to $\mathbb{C}$, where i and j are integers with $1 \le i \le s-1$ and $1 \le j \le r-1$.

3.3.8 LEMMA. *The characteristic polynomial of* ba^t *on* V_i

where t *and* i *are integers with* $0 \le t \le s-1$ *and* $1 \le i \le s-1$ *is*

$$(-1)^{r-1}\left[\vartheta^{r-1} - s\sum_{j=1}^{r-1} \lambda_i^{(r-j)t-1} c_{i,j}\vartheta^{j-1}\right] .$$

Proof. Firstly

$$\beta_{i,j}^{b} = \left(\lambda_i^{s-1}-1\right)\varepsilon_{1,j}^{b} + \ldots + \left(\lambda_i-1\right)\varepsilon_{s-1,j}^{b} .$$

From Lemma 3.3.4 *(ii)* if $j = 1, \ldots, r-2$ then $\varepsilon_{i,j}^{b} = \varepsilon_{i,j+1}$ so that $\beta_{i,j}^{b} = \beta_{i,j+1}$. However if $j = r-1$ then

$$\varepsilon_{i,j}^{b} = \sum_{l=1}^{r-1}\sum_{k=1}^{s-1} u_{k,l}\varepsilon_{k,l}^{a^{i-1}} = \sum_{l=1}^{r-1}\sum_{k=1}^{s-1} c_{k,l}\beta_{k,l}^{a^{i-1}} = \sum_{l=1}^{r-1}\sum_{k=1}^{s-1} c_{k,l}\lambda_k^{i-1}\beta_{k,l}$$

since $\beta_{k,l}^{a} = \lambda_k\beta_{k,l}$. Thus

$$\beta_{i,r-1}^{b} = \left(\lambda_i^{s-1}-1\right)\sum_{l=1}^{r-1}\sum_{k=1}^{s-1} c_{k,l}\beta_{k,l} + \left(\lambda_i^{s-2}-1\right)\sum_{l=1}^{r-1}\sum_{k=1}^{s-1} c_{k,l}\lambda_k\beta_{k,l} + \ldots$$

$$+ \left(\lambda_i-1\right)\sum_{l=1}^{r-1}\sum_{k=1}^{s-1} c_{k,l}\lambda_k^{s-2}\beta_{k,l}$$

$$= \sum_{l=1}^{r-1}\sum_{k=1}^{s-1} c_{k,l}\left[\left(\lambda_i^{s-1}-1\right) + \left(\lambda_i^{s-2}-1\right)\lambda_k + \ldots + \left(\lambda_i-1\right)\lambda_k^{s-2}\right]\beta_{k,l}$$

$$= \sum_{l=1}^{r-1} c_{i,l}\left(s\lambda_i^{s-1}\right)\beta_{i,l}$$

by Lemma 3.4.5. Consequently

$$\beta_{i,j}^{ba^t} = \beta_{i,j+1}^{a^t} = \lambda_i^t\beta_{i,j+1}$$

for $j = 1, \ldots, r-2$ and

$$\beta_{i,r-1}^{ba^t} = s\sum_{j=1}^{r-1} c_{i,j}\lambda_i^{s+t-1}\beta_{i,j} .$$

For convenience we put $d_j = sc_{i,j}$ here.

The characteristic polynomial of the action of ba^t on V_i is the determinant

$$\begin{vmatrix} -\vartheta & \lambda_i^t & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\vartheta & \lambda_i^t & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -\vartheta & \lambda_i^t & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\vartheta & \lambda_i^t \\ d_1\lambda_i^{t-1} & d_2\lambda_i^{t-1} & d_3\lambda_i^{t-1} & \cdots & d_{r-3}\lambda_i^{t-1} & d_{r-2}\lambda_i^{t-1} & d_{r-1}\lambda_i^{t-1}-\vartheta \end{vmatrix}$$

$$= \lambda_i^{(r-1)t} \begin{vmatrix} -\vartheta/\lambda_i^t & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\vartheta/\lambda_i^t & 1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -\vartheta/\lambda_i^t & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\vartheta/\lambda_i^t & 1 \\ d_1\lambda_i^{-1} & d_2\lambda_i^{-1} & d_3\lambda_i^{-1} & \cdots & d_{r-3}\lambda_i^{-1} & d_{r-2}\lambda_i^{-1} & d_{r-1}\lambda_i^{-1}-\vartheta/\lambda_i^t \end{vmatrix}$$

$$= \lambda_i^{(r-1)t}(-1)^{r-1}\left[\left(\vartheta/\lambda_i^t\right)^{r-1} - \sum_{j=1}^{r-1} d_j\lambda_i^{s-1}\left(\vartheta/\lambda_i^t\right)^{j-1}\right]$$

$$= (-1)^{r-1}\left[\vartheta^{r-1} - s\sum_{j=1}^{r-1} c_{i,j}\lambda_i^{(r-j)t-1}\vartheta^{j-1}\right] . \qquad //$$

Proof of Theorem 3.3.1

Let θ be an isomorphism of C onto E, where C and E belong to $X(r)$. Let $C = \mathrm{gp}\left(c,\ d;\ c^{d^r} = wc^{\varepsilon}w'\right)$ and

$E = \mathrm{gp}\left(e,\ f;\ e^{f^r} = ve^{\varepsilon'}v'\right)$ be θ-compatible standard presentations with standard vectors w and v respectively.

Suppose that s is the torsion number of C and of E and that $s \neq 0, 1$.

Let $M = \langle c \rangle^C$ and $N = \langle e \rangle^E$. It was shown in the proof of Theorem 3.1.5 that $M\theta = N$.

Let K and L denote the subgroups of C'/C'' and E'/E'' generated by the sets $\left\{c^{sd^{k-1}};\ 1 \leq k \leq r\right\}$ and $\left\{e^{sf^{k-1}};\ 1 \leq k \leq r\right\}$ respectively; by Lemma 3.3.7 we have that K and L are the fixed point sets of C'/C'' and E'/E'' under the action of the elements of M and N respectively. Thus if θ' is the induced isomorphism of C'/C'' onto E'/E'' then $K\theta' = L$. Consequently, θ induces an isomorphism of $P = (C'/C'')/K$ onto $Q = (E'/E'')/L$ and hence an isomorphism $\overline{\theta}$ of $\mathbb{C} \otimes P$ onto $\mathbb{C} \otimes Q$.

We remark that M/C' and N/E' are the torsion subgroups of C/C' and of E/E' respectively (see Lemma 3.1.3) and are cyclic of order s. Since the presentations of C and E above are θ-compatible the proof of the following lemma is obvious.

3.3.9 LEMMA. *There are integers k and m with $0 \leq m \leq s-1$, $1 \leq k \leq s-1$, $(k, s) = 1$ and elements g, h in E' such that*

$$c\theta = e^k g \text{ and } d\theta = fe^m h .$$

3.3.10 LEMMA. *For each $i = 1, 2, \ldots, s-1$ let P_i and Q_i denote the eigenspaces of $\mathbb{C} \otimes P$ and $\mathbb{C} \otimes Q$ corresponding to an eigenvalue λ_i of d and of f respectively. Let $\overline{\theta}$ denote the induced isomorphism of $\mathbb{C} \otimes P$ onto $\mathbb{C} \otimes Q$. Then $P_i\overline{\theta} = Q_{i'}$, where*

i' is given by $\lambda_{i'}^{k} = \lambda_i$, and the characteristic polynomials of the action of dc^t on P_i and of fe^{tk+m} on $Q_{i'}$ are the same for $t = 0, 1, \ldots, s-1$.

Proof. If v belongs to P_i then $v^c = \lambda_i v$. Thus $(v^c)\overline{\theta} = (\lambda_i v)\overline{\theta}$ so that $(v\overline{\theta})^{c\theta} = \lambda_i(v\overline{\theta})$, and using Lemma 3.4.9 we have $(v\overline{\theta})^{e^k} = \lambda_i(v\overline{\theta})$, since $(v\overline{\theta})^{e^k g} = (v\overline{\theta})^{e^k}$. Now $(k, s) = 1$ so that there is an integer l with $kl \equiv 1$ modulo s. Thus $(v\overline{\theta})^{e} = (v\overline{\theta})^{e^{kl}} = \lambda_i^l(v\overline{\theta})$. But $\lambda_i^l = \lambda_{i'}$ where i' is given in the statement of the lemma so that $v\overline{\theta}$ belongs to $Q_{i'}$. However $\overline{\theta}$ is an isomorphism and since (Lemma 3.3.6) both P_i and $Q_{i'}$ have the same dimension we conclude that $P_i\overline{\theta} = Q_{i'}$.

Now if v belongs to P_i then $\left(v^{dc^t}\right)\overline{\theta} = (v\overline{\theta})^{(dc^t)\theta}$. However, by Lemma 3.3.9 we have $(dc^t)\theta = fe^m h.(e^k g)^t = fe^{kt+m}g'$, where g' belongs to $Q_{i'}$. Thus $\left(v^{dc^t}\right)\overline{\theta} = (v\overline{\theta})^{fe^{kt+m}}$. As $\overline{\theta}$ is an isomorphism we conclude that the characteristic polynomials of the action of dc^t on P_i and of fe^{kt+m} on $Q_{i'}$ are the same. //

Now let $a_{i,j} = 1/s \sum_{n=1}^{s-1} \lambda_i^n w_{n,j}$ and $b_{i,j} = 1/s \sum_{n=1}^{s-1} \lambda_i^n v_{n,j}$ where i and j are integers with $1 \le i \le s-1$ and $1 \le j \le r-1$. Using Lemma 3.3.5 we see that

$$w_{n,j} = \sum_{i=1}^{s-1} a_{i,j}\left(\lambda_i^{s-n}-1\right) \quad \text{and} \quad v_{n,j} = \sum_{i=1}^{s-1} b_{i,j}\left(\lambda_i^{s-n}-1\right) . \qquad (3.4)$$

By Lemmas 3.3.10 and 3.3.8 the characteristic polynomials

$$(-1)^{r-1}\left[\vartheta^{r-1} - s\sum_{j=1}^{r-1} \lambda_i^{(r-j)t-1} a_{i,j}\vartheta^{j-1}\right]$$

and

$$(-1)^{r-1}\left[\vartheta^{r-1} - s\sum_{j=1}^{r-1} \lambda_{i'}^{(r-j)(tk+m)-1} b_{i',j}\vartheta^{j-1}\right]$$

are equal, where k and m are integers given in Lemma 3.3.9 and i' is given by $\lambda_{i'}^{k} = \lambda_i$.

It follows that for every i, j and t with $1 \le i \le s-1$, $1 \le j \le r-1$ and $0 \le t \le s-1$ we have

$$s\lambda_i^{(r-j)t-1} a_{i,j} = s\lambda_{i'}^{(r-j)(tk+m)-1} b_{i',j}$$

so that

$$a_{i,j} = \lambda_{i'}^{-k[(r-j)t-1]}\lambda_{i'}^{(r-j)(tk+m)-1} b_{i',j}\,.$$

Consequently, for any p with $0 \le p \le s-1$ we have

$$\sum_{i=1}^{s-1} \lambda_i^{s-p} a_{i,j} = \sum_{i=1}^{s-1} \lambda_{i'}^{s-[kp-(k-1)-(r-j)m]} b_{i',j}$$

$$= \sum_{i=1}^{s-1} \lambda_i^{s-[kp-(k-1)-(r-j)m]} b_{i,j}$$

since $(k, s) = 1$ entails that i' ranges from 1 to $s-1$ as i does.

But by (3.4) we have $w_{n,j} = \sum_{i=1}^{s-1} a_{i,j}\left(\lambda_i^{s-n}-1\right)$ and $v_{n,j} = \sum_{i=1}^{s-1} b_{i,j}\left(\lambda_i^{s-n}-1\right)$ for $n = 1, 2, \ldots, s-1$ and by definition $w_{0,j} = v_{0,j} = 0$. Now clearly $\sum_{n=1}^{s-1} w_{n,j} = -s\sum_{i=1}^{s-1} a_{i,j}$ and

$\sum_{n=1}^{s-1} v_{n,j} = -s \sum_{i=1}^{s-1} b_{i,j}$. Hence, for each $p = 0, 1, \ldots, s-1$ we have that

$$w_{p,j} = \sum_{i=1}^{s-1} a_{i,j}\left(\lambda_i^{s-p}-1\right) = \sum_{i=1}^{s-1} a_{i,j}\lambda_i^{s-p} - \sum_{i=1}^{s-1} a_{i,j}$$

$$= \sum_{i=1}^{s-1} \lambda_i^{s-[kp-(k-1)-(r-j)m]} b_{i,j} + 1/s \sum_{i=1}^{s-1} w_{i,j}$$

$$= v_{k(p-1)-(r-j)m+1,j} + 1/s\left(\sum_{i=1}^{s-1} w_{i,j} - \sum_{i=1}^{s-1} v_{i,j}\right),$$

which completes the proof of Theorem 3.3.1. //

3.4. An example

We show using Theorem 3.3.1 that $C = \mathrm{gp}\left(c,\ d;\ c^{d^3} = cc^{d^2}c^{d}\right)$ and $E = \mathrm{gp}\left(e,\ f;\ e^{f^3} = ee^{f}e^{f^2}\right)$ are not isomorphic groups.

Firstly, both C and E belong to $X(3)$, and s the torsion number of both C and E is 2 .

We have $\sigma_k(c, d) = \sigma_k(e, f) = 1$ for $k = 1, 2, 3$ whereas $\sigma_k\left(e, f^{-1}\right) = -1$ for $k = 2, 3$ and $\sigma_1\left(e, f^{-1}\right) = 1$. Thus if an isomorphism θ of C onto E is to exist then the presentations above are θ-compatible.

Now

$$c^{-d^2}c^{d^3} = c^{-d^2}cc^{d^2}c^{d} = c^{-d^2}c^{d}.c^{-d}c.c^{2d^2}.c^{-d^2}c^{d}$$

and

$$e^{f^2}e^{f^3} = e^{-f^2}ee^{f}e^{f^2} = e^{-f^2}e^{f}.e^{-f}e.e^{2f}.e^{-f}e^{f^2}$$

so that $W = (-1, -2)$ and $V = (1, 0)$ are the standard vectors obtained from the standard presentations above of C and E respectively.

But

$$1/2\left(\sum_{i=1} w_{i,2} - \sum_{i=1} v_{i,2}\right) = 1/2(-2-0) = -1$$

and

$$w_{1,2} - 1/2\left(\sum_{i=1} w_{i,2} - \sum_{i=1} v_{i,2}\right) = -2 - (-1) = -1 .$$

As neither $v_{0,2}$ nor $v_{1,2}$ is equal -1 we conclude from Theorem 3.3.1 that C and E cannot be isomorphic.

APPENDIX

THE DETERMINATION OF T-SYSTEMS BY THE CALCULATION OF FIBONACCI SEQUENCES ON A FINITE GROUP*

In this appendix we describe an empirical method for the determination of T-systems which is based on the calculation of Fibonacci sequences on a group. As examples we determine the T-systems of A_5 (the alternating group on 5 letters), S_5 (the symmetric group on 5 letters), SL(2, 5) (the group of 2×2 matrices, over Z_5, with determinant 1), PSL(2, 7) (the group of 2×2 matrices, over Z_7, with determinant ± 1), and A_6 (the alternating group on 6 letters). We note that A_5 and S_5 have already been shown to have 2 and 3 T-systems respectively by B.H. Neumann and Hanna Neumann in [14].

Let F denote the free group of rank 2 with generating pair (x, y). Let A and I denote, respectively, the automorphism group and the inner automorphism group of F.

Let G be a finite two generator group, $\Gamma(G)$ the set of generating pairs of G, and let B denote the automorphism group of G.

If $\mathfrak{g}AB$, where $\mathfrak{g}$ belongs to $\Gamma(G)$, is a T-system of G then referring to Lemma 2.2.2 we see that $\mathfrak{g}AB$ is the union of the sets $\mathfrak{g}B\eta$, where η is a representative of A modulo I.

---

* It has just come to our attention that Daniel Stork has also calculated the T-systems of A_6 and PSL(2, 7) in "The action of the automorphism group of F_2 upon the A_6 and PSL(2, 7)-defining subgroups of F_2", *Trans. Amer. Math. Soc.* 172 (1972), 111-117.

Now by Satz 6.7 of [14] the group A is generated, modulo I, by the elements μ and ρ given by

$$x\mu = y \ , \ \ y\mu = x$$

and

$$x\rho = x \ , \ \ y\rho = xy \ .$$

But if φ is the element of A given by

$$x\varphi = y \ , \ \ y\varphi = yx$$

then clearly $\varphi = \rho\mu$. It follows that A is generated, modulo I, by μ and φ.

We note that φ has the property, by Lemma 1.1.1, that $x\varphi^k = w_{k+1}$ for any integer k, where w_{k+1} is the $(k+1)$-st Fibonacci word. Also if $\mathrm{g} = (g_1, g_2)$ belongs to $\Gamma(G)$ and $f(g_1, g_2)$ is a Fibonacci sequence on G then

$$\mathrm{g}\varphi^k = (g_1, g_2)\varphi^k = \big(w_{k+1}(g_1, g_2), w_{k+2}(g_1, g_2)\big) = (f_{k+1}, f_{k+2}) \ .$$

Thus consecutive terms of the Fibonacci sequence $f(g_1, g_2)$ are the generating pairs $\mathrm{g}\varphi^k$.

We say that two generating pairs g and h of G are *B-equivalent* if there is an automorphism β of G with $\mathrm{g}\beta = \mathrm{h}$ and *B-inequivalent* otherwise.

Now let g be a generating pair of G and let $\mathrm{g}\langle\varphi\rangle$ denote the set $\{\mathrm{g}, \mathrm{g}\varphi, \ldots, \mathrm{g}\varphi^{r-1}\}$ where r is the first positive integer such that g and $\mathrm{g}\varphi^r$ are B-equivalent. Thus $\mathrm{g}\langle\varphi\rangle$ is a transitivity set under φ of B-inequivalent generating pairs. We remark that since $\mathrm{g}\varphi^r = (f_{r+1}, f_{r+2}) = \mathrm{g}\beta$ for some automorphism β of G we have $(f_{k+r+1}, f_{r+k+2}) = \mathrm{g}\varphi^{k+r} = (\mathrm{g}\varphi^k)\varphi^r = (f_{k+1}, f_{k+2})\beta$.

In particular this implies that the sequence of orders of the elements $f_n(g_1, g_2)$ is periodic of period r. This sequence of orders of the elements $f_n(g_1, g_2)$ provides each $g\langle\varphi\rangle$ with a convenient identification tag.

When dealing with a permutation representation of G the cycle structure of the elements $f_n(g_1, g_2)$ is periodic of period r. For example, if $g_1 = (12345)$ and $g_2 = (132)$ then g is a generating pair of A_5. Listing the elements of $f(g_1, g_2)$ we have

(12345), (132), (145), (14532), (15324), (13)(25), (12354),
(15432), (145), (132), (13245), (12453), (14)(25), (15423),
(12345), (132) .

Thus $g\varphi^{14} = g$. But clearly the cycle structure of the elements in this sequence begins to repeat after the 7-th terms, and in fact $g\varphi^7 = g\beta$ where β is the inner automorphism induced by the element (25)(34) .

When dealing with a matrix representation we can consider the sequence of traces of the elements $f_n(g_1, g_2)$. This is also periodic of period r.

We now summarize briefly the steps used in the determination of the T-systems of G.

(1) From a knowledge of the subgroup structure of G a formula of P. Hall [9] gives the number of generating pairs of G and dividing this number by the order of B we obtain the number of B-classes of G.

(2) A suitable representation (permutation or matrix) of G is found.

(3) A set Σ is constructed which is a union of transitivity sets of generating pairs under φ, and which contains a representative of every B-class of G. An *ad hoc* proceedure has to be used for its construction.

From (1) the order of Σ is known. Using our remarks above about identification tags it is not hard to find enough generating pairs of G to give an appropriate number of suitable transitivity sets under φ.

(4) For each $g = (g_1, g_2)$ in Σ we find the generating pairs $h = (h_1, h_2)$ in Σ such that the elements $f_n(g_2, g_1)$ and $f_n(h_1, h_2)$ of the Fibonacci sequences $f(g_2, g_1)$ and $f(h_1, h_2)$ respectively have the same orders. An h is then selected which is B-equivalent to $g\mu$. It follows that the transitivity sets under φ containing g and h are contained in the same T-system of G. In this way we find which B-class representatives lie in the same T-system.

Remarks. (i) Let $GL(2, p)$ be the group of 2×2 matrices with entries in Z_p, where p is a prime number, having non-zero determinant. If A and B belong to $GL(2, p)$ then

$$tr(BA) + det(A).tr(A^{-1}B) = tr(A).tr(B) ,$$

where $tr(X)$ denotes the trace of the matrix X and $det(X)$ denotes the determinant of X. Thus, given the determinants of two generators, their traces, and the trace of their product, we can calculate the sequence of traces (modulo p) of the elements of the Fibonacci sequence $f(A, B)$. If A and B belong to the subgroup $SL(2, p)$ of $GL(2, p)$ consisting of the matrices with determinant 1 then the formula takes the simple form

$$\operatorname{tr}(BA) + \operatorname{tr}\left(A^{-1}B\right) = \operatorname{tr}(A).\operatorname{tr}(B) .$$

(ii) It follows from Theorem 2.2.6 that for any generating pair $\mathrm{g} = (g_1, g_2)$ of G the order of $[g_1, g_2]$ is an invariant of the generating pairs in the T-system containing g. This is known as Higman's Criterion (see [15]) and is usually of use in part (4) above.

A_5

The number of B-inequivalent pairs is 19 .

Choose $\mathrm{g}_1 = ((12345), (254))$.

Then $\mathrm{g}_1\langle\varphi\rangle = \{((12345), (254)), ((254), (12)(34)), ((12)(34), (15432)), ((15432), (153)), ((153), (14352)), ((14352), (14)(25)), ((14)(25), (135)), ((135), (13254)), ((13254), (15432))\}$,

and $\mathrm{g}_1\varphi^{10} = ((14235), (134)) = \mathrm{g}_1\beta$, where β is an automorphism of order 5 (in fact an inner automorphism induced by (12354)) .

Thus $\mathrm{g}_1\varphi^{50} = \mathrm{g}_1$.

Choose $\mathrm{g}_2 = ((12345), (123))$.

Then $\mathrm{g}_2\langle\varphi\rangle = \{((12345), (123)), ((123), (13245))\}$,

and $\mathrm{g}_2\varphi^2 = ((13245), (245)) = \mathrm{g}_2\beta$, where β is an automorphism of order 6 (in fact an outer automorphism induced by the permutation (124)(35)) .

Thus $\mathrm{g}_2\varphi^{12} = \mathrm{g}_2$.

Choose $g_3 = \big((12345),\ (132)\big)$.

Then $g_3\langle\varphi\rangle = \big\{\big((12345),\ (132)\big),\ \big((132),\ (145)\big),\ \big((145),\ (14532)\big),$
$\big((14532),\ (15324)\big),\ \big((15324),\ (13)(25)\big),$
$\big((12354),\ (15432)\big)\big\}$,

and $g_3\varphi^7 = \big((15432),\ (145)\big) = g_3\beta$, where β is an automorphism of order 2 (in fact an inner automorphism induced by $(25)(34)$) .

Thus $g_3\varphi^{14} = g_3$.

I. $g_1\langle\varphi\rangle B$ forms one T-system of A_5 . A presentation associated with g_1 is

$$(x,\ y;\ x^5,\ y^3,\ (xy)^2) .$$

II. $g_2\langle\varphi\rangle B \cup g_3\langle\varphi\rangle B$ forms one T-system of A_5 . A presentation associated with g_2 is

$$(x,\ y;\ x^5,\ y^3,\ (x^2y)^2) .$$

S_5

The number of B-inequivalent pairs is 57 .

Choose $g_1 = \big((13425),\ (1523)\big)$.

Then $g_1\varphi^{18} = \big((14235),\ (1534)\big) = g_1\beta$, where β is an automorphism of order 3 (in fact an inner automorphism induced by (234)) .

Thus $g_1\varphi^{54} = g_1$.

Choose $g_2 = \{(2543), (143)(25)\}$.

Then $g_2\varphi^{15} = \{(1453), (253)(14)\} = g_2\beta$, where β is an automorphism of order 2 $\{$in fact an inner automorphism induced by (12)(54)$\}$.

Thus $g_2\varphi^{30} = g_2$.

Choose $g_3 = \{(12453), (135)(24)\}$.

Then $g_3\varphi^{24} = \{(13245), (123)(45)\} = g_3\beta$, where β is an automorphism of order 4 $\{$in fact an inner autormophism induced by (1245)$\}$.

Thus $g_3\varphi^{96} = g_3$.

I. $g_1\langle\varphi\rangle B$ forms one T-system of S_5 . A presentation associated with g_1 is

$$\{x, y; x^5, y^4, (xy)^2, (x^2y^2)^3\} .$$

II. $g_2\langle\varphi\rangle B$ forms one T-system of S_5 . A presentation associated with g_2 is

$$\{x, y; x^4, y^6, (xy)^2, (x^{-1}y)^3\} .$$

III. $g_3\langle\varphi\rangle B$ forms one T-system of S_5 . A presentation associated with g_3 is

$$\{x, y; x^5, y^6, (xy)^2, (x^2y^2)^2\} .$$

A_6

The number of B-inequivalent pairs is 53 .

Choose $g_1 = \{(135)(246), (14)(2635)\}$.

Then $g_1\varphi^7 = \{(146)(235), (24)(1635)\} = g_1\beta$, where β is an automorphism of order 2 .

Thus $g_1\varphi^{14} = g_1$.

Choose $g_2 = \{(23456), (16)(2345)\}$.

Then $g_2\varphi^5 = \{(13462), (36)(1524)\} = g_2\beta$, where β is an automorphism of order 2 .

Thus $g_2\varphi^{10} = g_2$.

Choose $g_3 = \{(154)(236), (24563)\}$.

Then $g_3\varphi^4 = \{(163)(254), (13624)\} = g_3\beta$, where β is an automorphism of order 5 .

Thus $g_3\varphi^{20} = g_3$.

Choose $g_4 = \{(12345), (246)\}$.

Then $g_4\varphi^{11} = \{(14365), (154)(263)\} = g_4\beta$, where β is an automorphism of order 2 .

Thus $g_4\varphi^{22} = g_4$.

Choose $g_5 = \{(25)(1364), (16325)\}$.

Then $g_5\varphi^{16} = \{(15)(2643), (13425)\} = g_5\beta$, where β is an automorphism of order 8 .

Thus $g_5\varphi^{128} = g_5$.

Choose $g_6 = \big((16543), (12345)\big)$.

Then $g_6\varphi^8 = \big((14263), (15324)\big) = g_6\beta$, where β is an

automorphism of order 10 .

Thus $g_6\varphi^{80} = g_6$.

Choose $g_7 = \big((12)(3645), (16245)\big)$.

Then $g_7\varphi^2 = \big((1436)(25), (15432)\big) = g_7\beta$, where β is an

automorphism of order 5 .

Thus $g_7\varphi^{10} = g_7$.

I. $g_1\langle\varphi\rangle B \cup g_2\langle\varphi\rangle B$ forms a T-system of A_6 . A presentation associated with g_1 is

$$(x, y;\ x^3, y^4, (xy)^5, [x, y]^2) .$$

II. $g_3\langle\varphi\rangle B \cup g_4\langle\varphi\rangle B$ forms a T-system of A_6 . A presentation associated with g_3 is

$$\big(x, y;\ x^3, y^5, (xy)^3, (xy^{-1})^4, [x, y^2]^2\big) .$$

III. $g_5\langle\varphi\rangle B$ forms a T-system of A_6 . A presentation associated with g_5 is

$$\big(x, y;\ x^4, y^5, (yx)^2, (yx^{-1})^5\big) .$$

IV. $g_6\langle\varphi\rangle B \cup g_7\langle\varphi\rangle B$ forms a T-system of A_6 . A presentation associated with g_6 is

$$\big(x, y;\ x^5, y^5, (xy)^2, (x^{-1}y)^4\big) .$$

SL(2, 5)

The number of B-inequivalent pairs is 76 .

Choose $g_1 = \left(\begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}\right)$.

Then $g_1\varphi^{21} = \left(\begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 4 & 3 \end{pmatrix}\right) = g_1\beta$, where β is an automorphism of order 2 .

Thus $g_1\varphi^{42} = g_1$.

Choose $g_2 = \left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}\right)$.

Then $g_2\varphi^{7} = \left(\begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right) = g_2\beta$, where β is an automorphism of order 2 .

Thus $g_2\varphi^{14} = g_2$.

Choose $g_3 = \left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}\right)$.

Then $g_3\varphi^{6} = \left(\begin{pmatrix} 4 & 0 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}\right) = g_3\beta$, where β is an automorphism of order 2 .

Thus $g_3\varphi^{12} = g_3$.

Choose $g_4 = \left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix}\right)$.

Then $g_4\varphi^{2} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 3 & 1 \end{pmatrix}\right) = g_4\beta$, where β is an automorphism of order 6 .

Thus $g_4\varphi^{12} = g_4$.

Choose $g_5 = \left(\begin{pmatrix}2 & 4\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 4\\ 1 & 0\end{pmatrix}\right)$.

Then $g_5\varphi^{30} = \left(\begin{pmatrix}3 & 4\\ 4 & 4\end{pmatrix}, \begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}\right) = g_5\beta$, where β is an automorphism of order 5 .

Thus $g_5\varphi^{150} = g_5$.

Choose $g_6 = \left(\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}, \begin{pmatrix}3 & 4\\ 4 & 4\end{pmatrix}\right)$.

Then $g_6\varphi^{10} = \left(\begin{pmatrix}1 & 4\\ 4 & 2\end{pmatrix}, \begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}\right) = g_6\beta$, where β is an automorphism of order 5 .

Thus $g_6\varphi^{50} = g_6$.

I. $g_1\langle\varphi\rangle B \cup g_2\langle\varphi\rangle B \cup g_3\langle\varphi\rangle B \cup g_4\langle\varphi\rangle B$ forms one T-system of SL(2, 5) . A presentation associated with the generating pair g_1 is

$$(x,\ y;\ x^6,\ y^5,\ x^{-3}(x^2y)^2) .$$

II. $g_5\langle\varphi\rangle B \cup g_6\langle\varphi\rangle B$ forms one T-system of SL(2, 5) . A presentation associated with the generating pair g_5 is

$$(x,\ y;\ x^5,\ y^6,\ x^{-3}(xy)^2) .$$

PSL(2, 7)

The number of B-inequivalent pairs is 57 .

Choose $g_1 = \left(\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}4 & 2\\ 5 & 1\end{pmatrix}\right)$.

Then $g_1\varphi^{14} = \left(\begin{pmatrix}5 & 3\\ 5 & 6\end{pmatrix}, \begin{pmatrix}6 & 5\\ 2 & 3\end{pmatrix}\right) = g_1\beta$, where β is an automorphism of order 3 .

Thus $g_1\varphi^{42} = g_1$.

Choose $g_2 = \left(\begin{pmatrix}4 & 0\\1 & 2\end{pmatrix}, \begin{pmatrix}3 & 1\\2 & 1\end{pmatrix}\right)$.

Then $g_2\varphi^4 = \left(\begin{pmatrix}1 & 4\\5 & 0\end{pmatrix}, \begin{pmatrix}6 & 3\\5 & 5\end{pmatrix}\right) = g_2\beta$, where β is an automorphism of order 6 .

Thus $g_2\varphi^{24} = g_2$.

Choose $g_3 = \left(\begin{pmatrix}0 & 6\\1 & 3\end{pmatrix}, \begin{pmatrix}0 & 6\\1 & 2\end{pmatrix}\right)$.

Then $g_3\varphi^2 = \left(\begin{pmatrix}6 & 4\\2 & 5\end{pmatrix}, \begin{pmatrix}4 & 2\\5 & 1\end{pmatrix}\right) = g_3\beta$, where β is an automorphism of order 4 .

Thus $g_3\varphi^8 = g_3$.

Choose $g_4 = \left(\begin{pmatrix}2 & 5\\6 & 5\end{pmatrix}, \begin{pmatrix}2 & 3\\6 & 6\end{pmatrix}\right)$.

Then $g_4\varphi^{14} = \left(\begin{pmatrix}2 & 5\\6 & 5\end{pmatrix}, \begin{pmatrix}6 & 4\\5 & 0\end{pmatrix}\right) = g_4\beta$, where β is an automorphism of order 2 .

Thus $g_4\varphi^{28} = g_4$.

Choose $g_5 = \left(\begin{pmatrix}0 & 6\\1 & 1\end{pmatrix}, \begin{pmatrix}5 & 0\\6 & 3\end{pmatrix}\right)$.

Then $g_5\varphi^5 = \left(\begin{pmatrix}2 & 5\\5 & 6\end{pmatrix}, \begin{pmatrix}6 & 5\\5 & 2\end{pmatrix}\right) = g_5\beta$, where β is an automorphism of order 2 .

Thus $g_5\varphi^{10} = g_5$.

Choose $g_6 = \left(\begin{pmatrix}0 & 2\\3 & 4\end{pmatrix}, \begin{pmatrix}5 & 0\\6 & 3\end{pmatrix}\right)$.

Then $g_6\varphi^2 = \left(\begin{pmatrix}0 & 3\\2 & 3\end{pmatrix}, \begin{pmatrix}4 & 2\\0 & 2\end{pmatrix}\right) = g_6\beta$, where β is an automorphism of order 7 .

Thus $g_6\varphi^{14} = g_6$.

Choose $g_7 = \left(\begin{pmatrix}5 & 2\\5 & 5\end{pmatrix}, \begin{pmatrix}6 & 4\\6 & 3\end{pmatrix}\right)$.

Then $g_7\varphi^{16} = \left(\begin{pmatrix}4 & 4\\4 & 6\end{pmatrix}, \begin{pmatrix}6 & 0\\6 & 6\end{pmatrix}\right) = g_7\beta$, where β is an automorphism of order 8 .

Thus $g_7\varphi^{128} = g_7$.

I. $g_1\langle\varphi\rangle B \cup g_2\langle\varphi\rangle B$ forms one T-system of PSL(2, 7) . A presentation associated with g_1 is

$$\langle x, y; x^4, y^7, (xy)^7, (x^{-1}y)^3, (xy^2)^2\rangle .$$

II. $g_3\langle\varphi\rangle B \cup g_4\langle\varphi\rangle B$ forms one T-system of PSL(2, 7) . A presentation associated with g_3 is

$$\langle x, y; x^2, y^3, (xy)^7, [x, y]^4\rangle .$$

III. $g_5\langle\varphi\rangle B \cup g_6\langle\varphi\rangle B$ forms one T-system of PSL(2, 7) . A presentation associated with g_5 is

$$\langle x, y; x^3, y^4, (xy)^3, (x^{-1}y)^4\rangle .$$

IV. $g_7\langle\varphi\rangle B$ forms one T-system of PSL(2, 7) . A presentation associated with g_7 is

$$\langle x, y; x^4, y^7, (xy)^2, (x^{-1}y)^3\rangle .$$

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